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Instability of Thick Elastic Solids

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1. Introduction

In this paper we investigate the stability of equilibrium of several homogeneous isotropic elastic solids. The problems considered are treated by making use of a non-linear three-dimensional theory of elasticity of the same sort as treated in the book of Green and Zerna [1], although the formulation of the theory used here follows that of Murnaghan [2]. No restrictive assumptions are made concerning the thickness of the bodies, and the displacements and strains are not assumed to be small. The strain energy function, from which the stress-strain relations come, could in principle be chosen arbitrarily; however, in this paper a special, but for many materials reasonable, choice for the strain energy function is made.

In each problem a simple solution is derived which satisfies all conditions of the exact non-linear theory. A small perturbation with respect to this simple state is then made, and the question is raised as to whether other possible equilibrium states — buckled states in other words — exist within the framework of the basic theory. Critical pressures and strains needed to cause such buckled solutions are found, together with the corresponding modes of buckling. It is observed that the critical buckling pressures and strains are the same as those furnished by the classical thin body theory in the limit case of small thickness.

Below we list a brief description of the problems considered.

Problem A: A circular cylinder is compressed along its curved lateral surface. The compression is such that the curved lateral surface of the undeformed cylinder goes into the curved lateral surface of a coaxial circular cylinder of smaller radius in such a way that no shear stress is developed. It is as though the cylinder were compressed by shrinking a very stiff greased ring down on it. The plane faces are assumed to be free of stress. We look for buckling in the axial direction.

Problem B: A hollow circular cylinder has a hydrostatic pressure applied to its outer curved lateral surface. The inner curved lateral surface is assumed to be free of stress. The axial displacement is taken to be zero. Thic can be brought about by requiring that the plane faces of the cylinder lie on rigid greased plane plates, the distance between the greased plates being the unstrained height of the cylinder. We look for buckling in the plane of the faces of the cylinder.

Problem C: A circular cylinder is subjected to axial compression. The compression is such that the end faces remain plane and horizontal and no shear stress is developed. This might be brought about by compressing the ends of the cylinder between two rigid plane horizontal greased plates. The curved lateral surface of the cylinder is assumed to be free of stress. We look for buckling perpendicular to the axis of the cylinder.

Problem D: This problem is the same as Problem C except that the cylinder is hollow and the inside surface is assumed to be free of stress.

Problem E: A hollow sphere has a hydrostatic pressure applied to its outer surface. Its inner surface is assumed to be free of stress.

Problems B and E have been treated previously by Lubkin [3] using somewhat different methods. The other problems have also been considered previously, but we believe not with the general non-linear theory used here.

2. The Non-Linear Theory

In this section we outline the theory to be used, largely without derivation. The notation used is that of Fritz John [4], [5].

Consider a rectangular Cartesian reference frame X. A particle which is at the point (x_1,x_2,x_3) when the body is unstrained will be at a point $(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ after the body is strained. The Eulerian coordinates $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ are treated as functions of the Lagrange coordinates x_1, x_2, and x_3. Dependence on time is ignored since we consider only equilibrium problems.

Let $p_{ij} = \dfrac{\partial \bar{x}_i}{\partial x_j}$ $(i,j = 1,2,3)$. Then W, the strain energy per unit undeformed volume, is a function of the quantities p_{ij}.

Let $q_{ij} = \dfrac{\partial W}{\partial p_{ij}}$ $(i,j = 1,2,3)$. If we neglect body forces, the equations of equilibrium are [5]

$$\frac{\partial q_{ij}}{\partial x_j} = 0 \,, \qquad i = 1,2,3 \tag{2.1}$$

where we use the usual summation convention.

Let S and $\overline{S}$ denote surfaces containing the same particles in the unstrained and strained bodies respectively. Let $\vec{n} = (n_1, n_2, n_3)$ and $\vec{\overline{n}} = (\overline{n}_1, \overline{n}_2, \overline{n}_3)$ be unit normal vectors pointing to corresponding sides of S and $\overline{S}$. Let $\vec{t} = (t_1, t_2, t_3)$ be the surface traction vector exerted on the surface $\overline{S}$ from the side to which $\vec{\overline{n}}$ points (the units of t_i are force per unit deformed area). Then

$$(2.2) \qquad t_i d\overline{S} = q_{ij} n_j dS \, , \qquad i = 1,2,3$$

where $d\overline{S}$ and dS are the elements of area of $\overline{S}$ and S respectively.

We can think of the transformation $d\overline{x}_i = p_{ij} dx_j$ as consisting of a rotation and a pure deformation which are defined as follows. Let p be the matrix (p_{ij}), and $c = p(\sqrt{p^* p})^{-1}$ ($*$ denotes the transpose and $\sqrt{p^* p}$ is the symmetric positive definite square root matrix of $p^* p$). Then c is orthogonal and $\det c > 0$ (we assume $\det p > 0$) so that c is a rotation matrix. For reasons stated later we call c the local rotation matrix of the deformation [6]. We define pure deformation matrices A and B by the equations $p = Ac = cB$ (i.e. p is the rotation followed by a pure deformation and p is a pure deformation followed by the rotation). We continue with a discussion of the rotation and pure deformation matrices.

Let ds and $d\overline{s}$ be the differentials of arc length along a curve consisting of the same particles in the undeformed and deformed bodies respectively. Then the eigenvalues of $p^* p$ and pp^* are the stationary values of $(\frac{d\overline{s}}{ds})^2$ with respect to variations of direction. The eigenvectors of $p^* p$ and pp^* give the

directions of stationary $\frac{d\bar{s}}{ds}$ in the undeformed and deformed bodies respectively. The matrix p carries each eigenvector of p^*p into an eigenvector of pp^*. Thus if z is a unit eigenvector of p^*p and λ is the corresponding stationary value of $\frac{d\bar{s}}{ds}$, then $\bar{z} = \frac{1}{\lambda}pz$ is a unit eigenvector of pp^*. Since $cz = p(\sqrt{p^*p})^{-1}z = \frac{1}{\lambda}pz = \bar{z}$, we see that c rotates the directions of stationary $\frac{d\bar{s}}{ds}$ in the undeformed body into the directions of stationary $\frac{d\bar{s}}{ds}$ in the deformed body. This is the reason we call c the local rotation matrix of the local deformation p.

Since $c^*p = \sqrt{p^*p}$, we see that c^*p is symmetric and positive definite. We next prove that these conditions characterize c. Let d be any other rotation matrix such that d^*p is symmetric and positive definite. Then $(d^*p)^2 = (d^*p)^*d^*p = p^*p$ so that d^*p is a square root matrix of p^*p. But there is only one symmetric positive definite square root matrix of p^*p, namely c^*p. Hence $d^*p = c^*p$ and $d = c$.

Similarly $C = (\sqrt{pp^*})^{-1}p$ is the unique rotation matrix such that pC^* is symmetric and positive definite. However $Cz = (\sqrt{pp^*})^{-1}pz = \lambda(\sqrt{pp^*})^{-1}\bar{z} = \bar{z}$ for all unit eigenvectors z of p^*p. Hence $C = c$ and c is also characterized by the condition that c is a rotation matrix and pc^* is symmetric and positive definite. Also $pc^* = \sqrt{pp^*}$.

From the previous definitions of the pure deformation matrices A and B we see that $A = pc^* = \sqrt{pp^*}$ and $B = c^*p = \sqrt{p^*p}$. Hence the eigenvalues of the pure deformation matrices are the

stationary values of $\frac{d\bar{s}}{ds}$, and the eigenvectors give the directions of stationary $\frac{d\bar{s}}{ds}$ in the deformed and undeformed bodies respectively.

Natural candidates for the strain matrix are the pure deformation matrices minus the identity. We define $\eta = \sqrt{p^*p} - 1$ to be the strain matrix.

We now list

(2.3) $cc^* = 1$, $\det c > 0$, and c^*p (or equivalently pc^*) is symmetric and positive definite.

(2.4) $\eta = \sqrt{p^*p} - 1 = c^*p - 1$.

The matrix c denotes a rotation around an axis through an angle which we call the local rotation angle. Let (r_1, r_2, r_3) be a unit vector along the axis of rotation and denote the local rotation angle by ψ. ψ is taken to be positive if the rotation c would cause a right handed screw to advance in the direction of (r_1, r_2, r_3). Then

(2.5) $c_{ij} = \delta_{ij} \cos\psi + r_i r_j (1 - \cos\psi) - e_{ijk} r_k \sin\psi$

where $e_{ijk} = \pm 1$ if i, j, k is an even or odd permutation of 1, 2, 3 and $e_{ijk} = 0$ if two subscripts are equal. In some problems considered here one can determine the quantities r_i by inspection and then obtain ψ as a function of the quantities p_{ij} using the symmetry of the matrix c^*p.

For an isotropic material the strain energy density function W should be taken as a symmetric function of the stationary values of $\frac{d\bar{s}}{ds}$. Hence for such a material W is a

function of s_1, s_2, s_3 where s_i is the sum of the i-th powers of the eigenvalues of η. If we let the square bracket denote the trace of a matrix, then

$$s_i = [\eta^i] = [(c^*p - 1)^i] . \tag{2.6}$$

To obtain an expression for the Lagrange stresses q_{ij} we derive expressions for the partial derivatives $\frac{\partial s_i}{\partial p_{jk}}$. Since $s_1 = [\eta] = [c^*p] - 3$, we have $\frac{\partial s_1}{\partial p_{ij}} = c_{ij} + [\frac{\partial c^*}{\partial p_{ij}} p]$. From (2.3) we obtain $\frac{\partial c^*}{\partial p_{ij}} = -c^* \frac{\partial c}{\partial p_{ij}} c^*$. Hence

$$[\frac{\partial c^*}{\partial p_{ij}} p] = -[c^* \frac{\partial c}{\partial p_{ij}} c^*p] = -[(c^* \frac{\partial c}{\partial p_{ij}} c^*p)^*] = -[p^*c \frac{\partial c^*}{\partial p_{ij}} c]$$

$$= -[(\frac{\partial c^*}{\partial p_{ij}} c)(p^*c)] = -[\frac{\partial c^*}{\partial p_{ij}} cc^*p] = -[\frac{\partial c^*}{\partial p_{ij}} p] .$$

Thus $[\frac{\partial c^*}{\partial p_{ij}} p] = 0$ and $\frac{\partial s_1}{\partial p_{ij}} = c_{ij}$.

Next

$$s_2 = [\eta^2] = [\eta^*\eta] = [(p^*c - 1)(c^*p - 1)]$$

$$= [p^*p] - 2[c^*p] + 3 = [p^*p] - 2s_1 - 3$$

so that $\frac{\partial s_2}{\partial p_{ij}} = 2(p_{ij} - c_{ij})$.

Finally

$$s_3 = [\eta^3] = [(p^*c - 1)(p^*p - 2c^*p + 1)]$$

$$= [p^*cp^*p] - 3[p^*p] + 3[c^*p] - 3 = [p^*cp^*p] - 3(s_2 + s_1 + 1) ,$$

and

$$\frac{\partial s_3}{\partial p_{ij}} = \frac{\partial}{\partial p_{ij}} [p^*cp^*p] - 6p_{ij} + 3c_{ij} .$$

The procedure for differentiating [p*cp*p] is similar to that for [c*p] and the result is $\frac{\partial}{\partial p_{ij}}$ [p*cp*p] $= 3p_{ir}c_{sr}p_{sj}$. Thus $\frac{\partial s_3}{\partial p_{ij}} = 3p_{ir}c_{sr}p_{sj} - 6p_{ij} + 3c_{ij}$. Since $q_{ij} = \frac{\partial W}{\partial s_k}\frac{\partial s_k}{\partial p_{ij}}$ we have therefore:

$$
(2.7) \qquad \begin{aligned} q_{ij} &= \left(\frac{\partial W}{\partial s_1} - 2\frac{\partial W}{\partial s_2} + 3\frac{\partial W}{\partial s_3}\right)c_{ij} \\ &\quad + 2\left(\frac{\partial W}{\partial s_2} - 3\frac{\partial W}{\partial s_3}\right)p_{ij} + 3\frac{\partial W}{\partial s_3}p_{ir}c_{sr}p_{sj} . \end{aligned}
$$

This derivation of the quantities q_{ij} was shown to the author by F. John; although this author is not aware that it appears in any of F. John's writings.

The special strain energy function used here from now on is what might be called the standard strain energy function:

$$
(2.8) \qquad W = \frac{\lambda}{2}s_1^2 + \mu s_2 .
$$

It corresponds to the classical strain energy density function when λ and μ are the Lamé constants. For this strain energy function, (2.7) becomes

$$
(2.7') \qquad q_{ij} = (\lambda s_1 - 2\mu)c_{ij} + 2\mu p_{ij}
$$

and the equilibrium equations for zero body force become

$$
(2.9) \qquad 2\mu\frac{\partial p_{ij}}{\partial x_j} + (\lambda s_1 - 2\mu)\frac{\partial c_{ij}}{\partial x_j} + \lambda c_{ij}\frac{\partial s_1}{\partial x_j} = 0 .
$$

3. Introduction of Tensor Methods

In the problems treated here we find it convenient to introduce tensor methods and work directly in curvilinear coordinates.

Let $\theta_i = \theta_i(x_1,x_2,x_3)$ $(i = 1,2,3)$ be the curvilinear coordinates, and let $g_{ij} = \frac{\partial x_r}{\partial \theta_i}\frac{\partial x_r}{\partial \theta_j}$ be the θ-components of the metric tensor. The g^{ij}'s are defined by $g^{ij}g_{jk} = \delta^i_k$, and the g_{ij}'s and g^{ij}'s are used in the usual manner to lower and raise indices of the θ-components of tensors.

In Table 1 the right hand column presents symbols representing the θ-components of tensors whose X-components are represented by the symbols in the left hand column. For example, $P_{ij} = \frac{\partial x_r}{\partial \theta_i}\frac{\partial x_r}{\partial \theta_j} p_{rs}$. We call special attention to the fact that the tensors considered have the items in the left hand column as X-components rather than as $\overline{X}$-components.

X-components	θ-components
$\overline{x}_i$	u_i
p_{ij}	P_{ij}
q_{ij}	Q_{ij}
n_i	N_i
$\overline{n}_i$	$\overline{N}_i$
t_i	T_i
c_{ij}	C_{ij}
r_i	R_i
e_{ijk}	ε_{ijk}

Table 1

To give physical meaning to some of the quantities introduced by Table 1, we observe the following. Consider the vectors $\vec{x} = (x_1, x_2, x_3)$, $\vec{\bar{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, $\vec{n}$, $\vec{\bar{n}}$, and $\vec{t}$. Let $\vec{g}_i = \frac{\partial \vec{x}}{\partial \theta_i}$ so that $\vec{g}_i$ is a tangent vector to the i-th curvilinear curve. Then $\vec{\bar{x}} = u^i \vec{g}_i$, $\vec{n} = N^i \vec{g}_i$, $\vec{\bar{n}} = \bar{N}^i \vec{g}_i$, and $\vec{t} = T^i \vec{g}_i$. Thus u^i, etc., are the components of the corresponding vectors with respect to the $\vec{g}_i$'s.

Also we observe that $P_{ij} = u_i\big|_j$ where $\big|_j$ denotes covariant differentiation with respect to θ_j and $\varepsilon_{ijk} = \sqrt{g}\, e_{ijk}$ where $g = \det\,(g_{ij})$.

In curvilinear coordinates some of the results of Section 2 become

(3.1) $\quad Q^{ij}\big|_j = 0$,

(3.2) $\quad T^i d\bar{S} = Q^{ij} N_j dS$,

(3.3) $\quad C_k{}^i C^{kj} = g^{ij}$, $\det\,(C_{ij}) > 0$, $(C^{ki} P_k{}^j)$ is symmetric and positive definite,

(3.5) $\quad C_{ij} = g_{ij} \cos\psi + R_i R_j (1 - \cos\psi) - \varepsilon_{ijk} R^k \sin\psi$,

(3.6) $\quad s_1 = C^{ji} P_{ji} - 3$, etc.,

(3.7) $$Q_{ij} = \left(\frac{\partial W}{\partial s_1} - 2\frac{\partial W}{\partial s_2} + 3\frac{\partial W}{\partial s_3}\right) C_{ij} + 2\left(\frac{\partial W}{\partial s_2} - 3\frac{\partial W}{\partial s_3}\right) P_{ij} + 3\frac{\partial W}{\partial s_3} P_{ir} C^{sr} P_{sj} ,$$

(3.7') $\quad Q_{ij} = (\lambda s_1 - 2\mu) C_{ij} + 2\mu P_{ij}$ for standard strain energy ,

(3.9) $\quad 2\mu P^{ij}\big|_j + (\lambda s_1 - 2\mu)C^{ij}\big|_j + \lambda C^{ij}\dfrac{\partial s_1}{\partial\theta_j} = 0$ for standard strain energy.

4. The Perturbed Problem

In each of the problems considered we assume that the buckled solution can be expressed as a function of a parameter $\delta \geq 0$ such that the buckled solution coincides with the simple solution for $\delta = 0$ but differs from the simple solution for $\delta > 0$. We also assume δ can be chosen so that quantities depending on δ have a derivative with respect to δ at $\delta = 0$. If A is any such quantity, we let $\mathring{A}$ be the value of A for $\delta = 0$ and $\dot{A}$ the derivative with respect to δ at $\delta = 0$.

In the perturbed problem we treat the $\dot{u}^i$'s as the unknown functions. The main object of this section is to derive differential equations for the $\dot{u}^i$'s.

First of all from $P_{ij} = u_i\big|_j$ we obtain

(4.1) $$\dot{P}_{ij} = \dot{u}_i\big|_j \ .$$

From (3.3) we have

(4.2) $$\mathring{C}_k{}^i\dot{C}^{kj} + \dot{C}^{ki}\mathring{C}_k{}^j = 0 \ ,$$

(4.3) $$(\dot{C}^{ki}\mathring{P}_k{}^j + \mathring{C}^{ki}\dot{P}_k{}^j) \text{ is symmetric .}$$

Since the left side of equations (4.2) are the elements of a symmetric matrix, (4.2) can be thought of as six linear equations for the $\dot{C}^{ij}$'s. Condition (4.3) gives us three more linear

equations for the $\dot{C}^{ij}$'s, so (4.2) and (4.3) together determine the $\dot{C}^{ij}$'s as functions of the $\dot{P}^{ij}$'s and hence as functions of the $\dot{u}^{i}$'s through (4.1).

Next we derive

$$(4.4) \qquad \begin{cases} \dot{s}_1 = \mathring{C}_{ij}\dot{P}^{ij} \\ \dot{s}_2 = 2(\mathring{P}_{ij} - \mathring{C}_{ij})\dot{P}^{ij} \\ \dot{s}_3 = 3(\mathring{C}_{ij} - 2\mathring{P}_{ij} + \mathring{P}_{i\ell}\mathring{C}^{k\ell}P_{kj})\dot{P}^{ij} \end{cases}$$

which give the $\dot{s}_i$'s in terms of the $\dot{u}^{i}$'s.

Since $s_1 = [c*p]-3$, we have $\dot{s}_1 = [\mathring{c}*\dot{p}] + [\dot{c}*\mathring{p}]$. We can show that $[\dot{c}*\mathring{p}] = 0$ in the same way we showed $\frac{\partial}{\partial p_{ij}}[c*p] = 0$ in Section 2. Hence $\dot{s}_1 = \mathring{c}_{ij}\dot{p}_{ij} = \mathring{C}_{ij}\dot{P}^{ij}$.

Since $s_2 = [p*p] - 2s_1 - 3$, we have

$$\dot{s}_2 = 2\mathring{p}_{ij}\dot{p}_{ij} - 2\mathring{c}_{ij}\dot{p}_{ij} = 2(\mathring{P}_{ij} - \mathring{C}_{ij})\dot{P}^{ij} .$$

Since $s_3 = [p*cp*p] - 3(s_2 + s_1 + 1)$, we have

$$\dot{s}_3 = [\mathring{p}*\dot{c}\,\mathring{p}*\mathring{p}] + 3[\mathring{p}\,\mathring{c}*\mathring{p}\dot{p}^*] - 3(2\mathring{p}_{ij} - \mathring{c}_{ij})\dot{p}_{ij} .$$

Methods similar to those used to show $[\dot{c}*\mathring{p}] = 0$ can be used to show that $[\mathring{p}*\dot{c}\,\mathring{p}*\mathring{p}] = 0$. Thus $\dot{s}_3$ is given by (4.4).

By dotting (3.1) and (3.7) we now obtain the differential equations for the $\dot{u}^{i}$'s and the $\dot{Q}^{ij}$'s. Since we are using the standard strain energy function in our problems, we list these results only for that special case.

$$\dot{Q}^{ij} = (\lambda \mathring{s}_1 - 2\mu)\dot{C}^{ij} + \lambda \dot{s}_1 \mathring{C}^{ij} + 2\mu \dot{P}^{ij} , \tag{4.5}$$

$$\begin{aligned} 2\mu \dot{P}^{ij}\big|_j &+ (\lambda \mathring{s}_1 - 2\mu)\dot{C}^{ij}\big|_j \\ &+ \lambda \frac{\partial \mathring{s}_1}{\partial \theta_j} \dot{C}^{ij} + \lambda \frac{\partial \dot{s}_1}{\partial \theta_j} \mathring{C}^{ij} + \lambda \dot{s}_1 \mathring{C}^{ij}\big|_j = 0 . \end{aligned} \tag{4.6}$$

5. Special Curvilinear Coordinates

In the problems treated here either cylindrical or spherical coordinates are used. In this section we list some results for these two curvilinear coordinate systems.

For cylindrical coordinates we have $x_1 = \theta_1 \cos \theta_2$, $x_2 = \theta_1 \sin \theta_2$, $x_3 = \theta_3$. For convenience we write $r = \theta_1$, $\theta = \theta_2$, $z = \theta_3$. Then

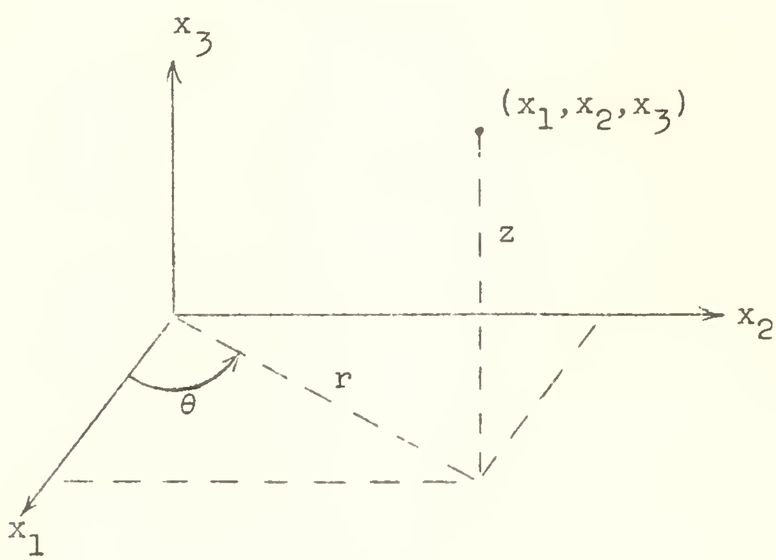


Fig. 1

$$
(5.1)\qquad \begin{cases} (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\[2ex] (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\[2ex] \quad g = r^2 \end{cases}
$$

The non-zero Christoffel symbols are

$$
(5.2)\qquad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r}\,, \qquad \Gamma^{1}_{22} = -r\;.
$$

Since

$$
P^{i}{}_{j} = u^{i}\big|_{j} = \frac{\partial u^{i}}{\partial \theta_{j}} + \Gamma^{i}_{kj} u^{k}
$$

and

$$
P^{ij}\big|_{j} = \frac{\partial P^{ij}}{\partial \theta_{j}} + \Gamma^{i}_{kj} P^{kj} + \Gamma^{j}_{kj} P^{ik}\,,
$$

we have

$$
(5.3)\qquad (P^{i}{}_{j}) = \begin{pmatrix} \dfrac{\partial u^1}{\partial r} & \dfrac{\partial u^1}{\partial \theta} - r\,u^2 & \dfrac{\partial u^1}{\partial z} \\[2ex] \dfrac{\partial u^2}{\partial r} + \dfrac{1}{r}\,u^2 & \dfrac{\partial u^2}{\partial \theta} + \dfrac{1}{r}\,u^1 & \dfrac{\partial u^2}{\partial z} \\[2ex] \dfrac{\partial u^3}{\partial r} & \dfrac{\partial u^3}{\partial \theta} & \dfrac{\partial u^3}{\partial z} \end{pmatrix}
$$

$$
(5.4)\qquad \begin{cases} P^{1j}\big|_{j} = \dfrac{\partial P^{1j}}{\partial \theta_{j}} + \dfrac{1}{r}\,P^{11} - rP^{22} \\[2ex] P^{2j}\big|_{j} = \dfrac{\partial P^{2j}}{\partial \theta_{j}} + \dfrac{1}{r}\,(P^{12} + 2P^{21}) \\[2ex] P^{3j}\big|_{j} = \dfrac{\partial P^{3j}}{\partial \theta_{j}} + \dfrac{1}{r}\,P^{31} \end{cases}
$$

For spherical coordinates we have $x_1 = \theta_1 \sin \theta_2 \cos \theta_3$, $x_2 = \theta_1 \sin \theta_2 \sin \theta_3$, $x_3 = \theta_1 \cos \theta_2$. For convenience we write $r = \theta_1$, $\theta = \theta_2$, $\phi = \theta_3$. Then

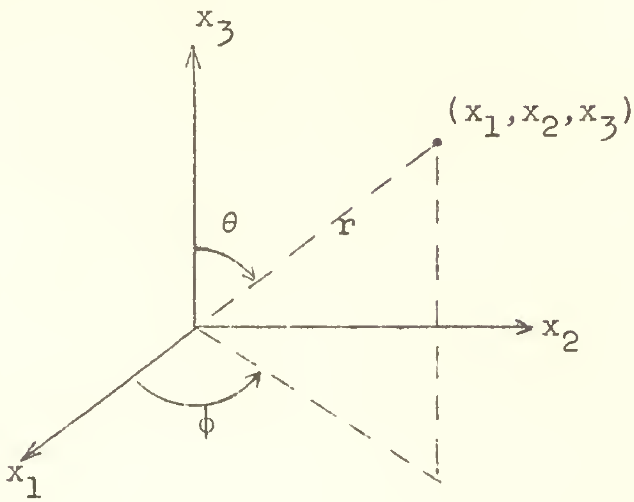


Fig. 2

$$\begin{cases} (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \\ (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \\ g = r^4 \sin^2\theta \end{cases}$$

The non-zero Christoffel symbols are

$$(5.6) \quad \begin{cases} \Gamma^1_{22} = -r \, , \quad \Gamma^1_{33} = -r \sin^2\theta \, , \quad \Gamma^2_{33} = -\sin\theta \cos\theta \, , \\ \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \, , \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \, , \end{cases}$$

also

$$(5.7)\quad (P^i{}_j) = \begin{pmatrix} \dfrac{\partial u^1}{\partial r} & \dfrac{\partial u^1}{\partial \theta} - ru^2 & \dfrac{\partial u^1}{\partial \phi} - ru^3 \sin^2\theta \\ \dfrac{\partial u^2}{\partial r} + \dfrac{1}{r}\, u^2 & \dfrac{\partial u^2}{\partial \theta} + \dfrac{1}{r}\, u^1 & \dfrac{\partial u^2}{\partial \phi} - u^3 \sin\theta \cos\theta \\ \dfrac{\partial u^3}{\partial r} + \dfrac{1}{r}\, u^3 & \dfrac{\partial u^3}{\partial \theta} + u^3 \cot\theta & \dfrac{\partial u^3}{\partial \phi} + \dfrac{1}{r}\, u^1 + u^2 \cot\theta \end{pmatrix}$$

$$(5.8)\quad \begin{cases} P^{1j}\big|_j = \dfrac{\partial P^{1j}}{\partial \theta_j} + \dfrac{2}{r}\, P^{11} - rP^{22} - rP^{33} \sin^2\theta + P^{12} \cot\theta \\ P^{2j}\big|_j = \dfrac{\partial P^{2j}}{\partial \theta_j} + P^{22} \cot\theta - P^{33} \sin\theta \cos\theta + \dfrac{1}{r}\,(P^{12} + 3P^{21}) \\ P^{3j}\big|_j = \dfrac{\partial P^{3j}}{\partial \theta_j} + \dfrac{1}{r}\,(P^{13} + 3P^{31}) + (P^{23} + 2P^{32}) \cot\theta \end{cases}$$

6. Problem A

This problem was treated earlier by this writer [7] as a separate problem. It seems advisable to include it here since this treatment is considerably simpler.

Cylindrical coordinates r, θ, z are used. The unstrained circular cylinder occupies the region $r \leq R$, $|z| \leq h$.

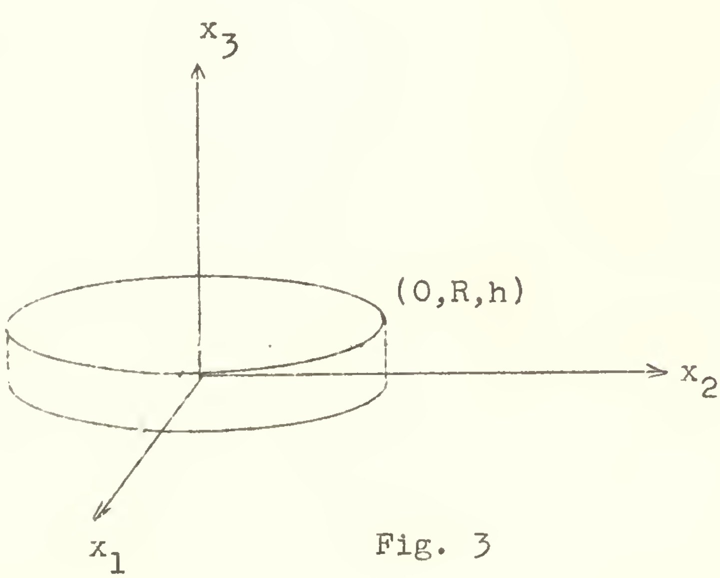


Fig. 3

We observe that $\vec{g}_1 = (\cos\theta, \sin\theta, 0)$, $\vec{g}_2 = r(-\sin\theta, \cos\theta, 0)$, and $\vec{g}_3 = (0,0,1)$ (see Section 3). Since $\vec{g}_2$ and $\vec{g}_3$ are perpendicular to the radial direction, the condition that the curved lateral boundary goes into the curved lateral boundary of a coaxial cylinder with smaller radius may be written as $u^1 = (1-\varepsilon)R$ for $r = R$ where ε is a positive constant. Since $\vec{g}_2$ and $\vec{g}_3$ are tangent to the deformed curved lateral surface, the condition that no shear stress is developed for $r = R$ becomes $T^2 = T^3 = 0$. Hence the boundary conditions can be written as

$$
(6.1) \qquad \begin{cases} \left.\begin{aligned} T^2 &= T^3 = 0 \\ u^1 &= (1-\varepsilon)R \end{aligned}\right\} & \text{for } r = R \\ T^i = 0 \quad (i = 1,2,3) & \text{for } z = \pm h \end{cases}
$$

On the boundary $r = R$, $N_1 = 1$, $N_2 = N_3 = 0$ and on $z = \pm h$, $N_1 = N_2 = 0$, $N_3 = \pm 1$. From (3.2) we see that the boundary conditions can be rewritten as

$$
(6.2) \qquad \begin{cases} \left.\begin{aligned} Q^{21} &= Q^{31} = 0 \\ u^1 &= (1-\varepsilon)R \end{aligned}\right\} & \text{for } r = R \\ Q^{i3} = 0 \quad (i = 1,2,3) & \text{for } z = \pm h \end{cases}
$$

We look for a simple solution in which the cylinder is linearly extended in the radial and axial directions with no rotation around the axis of the cylinder. Since $\vec{\vec{x}} = u^i \vec{g}_i$, this means $u^1 = ar$, $u^2 = 0$, $u^3 = bz$ for the simple solution where a and b are constants.

From (5.3) we have

$$(P^i{}_j) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \text{ and hence } (P^{ij}) = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{a}{r^2} & 0 \\ 0 & 0 & b \end{pmatrix} .$$

Since (P^{ij}) is symmetric and positive definite (for $a,b > 0$), we have $C^{ij} = g^{ij}$ from (3.3).

From (3.6) we have $s_1 = 2a + b - 3$.

Substituting P^{ij}, C^{ij}, and s_1 into (3.9), we see that the simple solution satisfies the equilibrium equations.

From (3.7') we see that (Q^{ij}) is diagonal,

$$Q^{11} = 2(\lambda+\mu)a - 3\lambda - 2\mu \ , \quad Q^{22} = \frac{1}{r^2} Q^{11} \ ,$$

and

$$Q^{33} = 2\lambda a + (\lambda + 2\mu)b - 3\lambda - 2\mu \ .$$

Substituting into (6.2) we obtain

$$\text{(6.3)} \qquad \begin{cases} a = 1-\varepsilon \\ b = 1 + \dfrac{2\lambda\varepsilon}{\lambda + 2\mu} = 1 + \dfrac{2\sigma\varepsilon}{1-\sigma} \end{cases}$$

where σ is Poisson's ratio.

Thus we observe there is a simple solution of the type considered when $0 \leq \varepsilon < 1$.

On the boundary $r = R$, $d\overline{S} = abr\, dr d\theta$ and $dS = r\, dr d\theta$. Hence letting T be the normal pressure per unit deformed area applied to the edge of the plate, we obtain

$$T = -T^1 = -Q^{11}\frac{dS}{d\overline{S}} = \frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu}\ \frac{\varepsilon}{ab} = \frac{E\varepsilon}{(1-\varepsilon)(1-\sigma+2\sigma\varepsilon)}$$

where E is Young's modulos. Hence $T \longrightarrow \infty$ as the radius of the deformed cylinder goes to zero (i.e. as $\varepsilon \longrightarrow 1$).

If we let P be the total load applied to the edge of the deformed cylinder, we have $P = 4\pi abRhT = 4\pi Rh\varepsilon \frac{E}{1-\sigma}$. Hence the total load approaches a finite limit as the deformed radius goes to zero. Also, as the radius goes to zero, the deformed height 2bh approaches a finite limit. Hence a finite total load shrinks the cylinder down to a line segment of finite length. From these considerations we see that the standard strain energy function does not give a good description of known materials unless the strains are not too large. However, no limitations on the magnitudes of strains are assumed in the following.

We look for special buckled solutions which are axially symmetric, and therefore are free of displacement in the θ-direction. Hence we take $u^1 = u^1(r,z,\delta)$, $u^2 = 0$, $u^3 = u^3(r,z,\delta)$. We write $u^1 = ar + V^1$, $u^3 = bz + V^3$. From (6.3) a and b are functions of ε which in turn is a function of the perturbation parameter δ. Hence $\dot{u}^1$ and $\dot{u}^3$ (see Section 4 for the meaning of $^\circ$ and $^\cdot$) have the form

$$(6.4) \qquad \begin{cases} \dot{u}^1 = \dot{a}r + \sum_{n=0}^{\infty} a_n(z)\alpha_n(r) \\ \dot{u}^3 = \dot{b}z + \sum_{n=0}^{\infty} b_n(z)\beta_n(r) \end{cases}$$

where we have used series for $\dot{V}^1$ and $\dot{V}^3$. That is, we express $\dot{u}^1$ and $\dot{u}^3$ as the perturbed simple solution plus a sum of products. We will attempt to choose the α_n's and β_n's so that variables separate when $\dot{u}^1$ and $\dot{u}^3$ are substituted into (4.6).

From (5.3), (4.2), (4.3), and (4.4) we have (remember $\overset{\circ}{C}{}^{ij} = g^{ij}$, $\overset{\circ}{C}_{ij} = g_{ij}$, $\overset{\circ}{C}{}^{i}{}_{j} = \overset{\circ}{C}{}_{j}{}^{i} = \delta^i_j$)

$$
(6.5)\quad \begin{cases}
(\dot{P}^{ij}) = \begin{pmatrix} \dfrac{\partial \dot{u}^1}{\partial r} & 0 & \dfrac{\partial \dot{u}^1}{\partial z} \\ 0 & \dfrac{1}{r^3}\,\dot{u}^1 & 0 \\ \dfrac{\partial \dot{u}^3}{\partial r} & 0 & \dfrac{\partial \dot{u}^3}{\partial z} \end{pmatrix} \\
\dot{C}^{12} = \dfrac{1}{2\overset{\circ}{a}}\,(\dot{P}^{12} - \dot{P}^{21}) = 0 \\
\dot{C}^{13} = \dfrac{1}{\overset{\circ}{a} + \overset{\circ}{b}}\,(\dot{P}^{13} - \dot{P}^{31}) \\
\dot{C}^{23} = \dfrac{1}{\overset{\circ}{a} + \overset{\circ}{b}}\,(\dot{P}^{23} - \dot{P}^{32}) = 0 \\
\overset{\circ}{s}_1 = 2\overset{\circ}{a} + \overset{\circ}{b} - 3 \\
\dot{s}_1 = \dfrac{\partial \dot{u}^1}{\partial r} + \dfrac{1}{r}\,\dot{u}^1 + \dfrac{\partial \dot{u}^3}{\partial z}
\end{cases}
$$

From (5.4), (6.5), and (6.4)

$$
\dot{P}^{1j}\big|_j = \sum [a_n(\alpha_n'' + \frac{1}{r}\,\alpha_n' - \frac{1}{r^2}\,\alpha_n) + a_n''\alpha_n] ,
$$

and

$$
\dot{P}^{3j}\big|_j = \sum [b_n(\beta_n'' + \frac{1}{r}\,\beta_n') + b_n''\beta_n] .
$$

To separate the variables in (4.6) we try to choose α_n and β_n so that the variables separate in $\dot{P}^{1j}\big|_j$ and $\dot{P}^{3j}\big|_j$ themselves, i.e. we choose α_n and β_n so that $\alpha_n'' + \frac{1}{r}\,\alpha_n' - \frac{1}{r^2}\,\alpha_n$ and α_n are proportional and so that $\beta_n'' + \frac{1}{r}\,\beta'$ and β_n are proportional. This can be accomplished by choosing

$$
(6.6)\quad \begin{cases} \alpha_n = J_1(k_n r) \\ \beta_n = J_o(k_n r) \end{cases}
$$

where J_o and J_1 are the usual Bessel functions, $k_o = 0$ and $k_n < k_{n+1}$ for $n = 0,1,2,\ldots$ are constants to be chosen at our convenience.

Since $k_o = 0$, we see that $\alpha_o = 0$ and $\beta_o = 1$. It follows from (6.4) that a_o is arbitrary. For convenience we choose $a_o = 0$.

We now have the following relations for α_n and β_n:

$$(6.7) \qquad \begin{cases} \alpha_n'' + \frac{1}{r}\alpha_n' + (k_n^2 - \frac{1}{r^2})\alpha_n = 0 , \\ \beta_n'' + \frac{1}{r}\beta_n' + k_n^2\beta_n = 0 , \\ \alpha_n' + \frac{1}{r}\alpha_n = k_n\beta_n , \\ \beta_n' = -k_n\alpha_n . \end{cases}$$

The first two equations are the differential equations for J_1 and J_o and the last two are familiar recurrence formulas for J_o and J_1. These can be used to replace derivatives of α_n and β_n by expressions involving α_n and β_n themselves.

Let $c_n = a_n' + k_n b_n$ and $d_n = b_n' + k_n a_n$. Using (6.5), (6.4) and (6.7) we have

$$\dot{C}^{13} = \frac{1}{\dot{a} + \dot{b}} \sum c_n\alpha_n$$

and

$$\dot{s}_1 = 2\dot{a} + \dot{b} + \sum d_n\beta_n .$$

In the same way it follows that

$$\dot{P}^{1j}\Big|_j = \sum (c_n' - k_n d_n)\alpha_n \quad ,$$

$$\dot{P}^{3j}\Big|_j = \sum (d_n' - k_n c_n)\beta_n \quad ,$$

$$\dot{C}^{1j}\Big|_j = \frac{1}{\overset{\circ}{a} + \overset{\circ}{b}} \sum c_n'\alpha_n \quad ,$$

$$\dot{C}^{3j}\Big|_j = -\frac{1}{\overset{\circ}{a} + \overset{\circ}{b}} \sum k_n c_n \beta_n \ .$$

Next observe that $\lambda\overset{\circ}{s}_1 - 2\mu = -2\mu\overset{\circ}{b}$ and substitute the above into (4.6). The variables separate, and we obtain the following ordinary differential equations.

$$\text{(6.8)} \qquad \left\{ \begin{aligned} Bc_n' - (\lambda + 2\mu)k_n d_n &= 0 \\ (\lambda + 2\mu)d_n' - Bk_n c_n &= 0 \end{aligned} \right\} \qquad n = 0,1,2,\ldots$$

where $B = \dfrac{2\mu\overset{\circ}{a}}{\overset{\circ}{a} + \overset{\circ}{b}}$.

Since $k_o = 0$, then $c_o = a_o' = 0$. Also (6.8) implies $d_o' = 0$ so that $d_o = A_o =$ constant.

For $n \geq 1$ the general solution of (6.8) is

$$c_n = (\lambda + 2\mu)(A_n \cosh k_n z + B_n \sinh k_n z)$$

$$d_n = B(A_n \sinh k_n z + B_n \cosh k_n z)$$

where A_n and B_n are constants.

Next integrating the system

$$a_n' + k_n b_n = c_n$$

$$b_n' + k_n a_n = d_n$$

we obtain $b_o = A_o z + B_o$ and for $n \geq 1$,

$$a_n = (\tfrac{A}{2} B_n z + C_n)\sinh k_n z + (\tfrac{A}{2} A_n z + D_n)\cosh k_n z \ ,$$

$$b_n = \left[-\frac{A}{2} A_n z - D_n + \frac{A+2B}{2k_n} B_n\right]\sinh k_n z$$

$$+ \left[-\frac{A}{2} B_n z - C_n + \frac{A+2B}{2k_n} A_n\right]\cosh k_n z$$

where $A = \lambda + \frac{2\mu\overset{\circ}{b}}{\overset{\circ}{a} + \overset{\circ}{b}}$ and the C_n's and D_n's are additional constants.

Next we turn to the boundary conditions. From (6.2) we see that the perturbed boundary conditions are

$$(6.9) \qquad \begin{cases} \left.\begin{aligned} \dot{Q}^{21} &= \dot{Q}^{31} = 0 \\ \dot{u}^{1} &= -\dot{\varepsilon}R \end{aligned}\right\} & \text{for } r = R \\ \dot{Q}^{i3} = 0 \quad (i = 1,2,3) & \text{for } z = \pm h \end{cases}$$

Using (4.5) and (6.5) the non-trivial boundary conditions become

$$\dot{Q}^{31} = 2\mu \sum \left(\frac{\overset{\circ}{b}}{\overset{\circ}{a} + \overset{\circ}{b}} c_n - k_n b_n\right)\alpha_n = 0 \quad \text{for } r = R \ ,$$

$$\dot{u}^{1} = \dot{a}R + \sum a_n\alpha_n = -\dot{\varepsilon}R \quad \text{for } r = R \ ,$$

$$\dot{Q}^{13} = \sum (Bc_n - 2\mu k_n b_n)\alpha_n = 0 \quad \text{for } z = \pm h \ ,$$

$$\dot{Q}^{33} = 2\lambda\dot{a} + (\lambda + 2\mu)\dot{b} + \sum [(\lambda + 2\mu)d_n - 2\mu k_n a_n]\beta_n = 0 \quad \text{for } z = \pm h \ .$$

Next we choose k_n so that $k_n R$ is the n-th positive zero of J_1, and we observe from (6.3) that $\dot{a} = -\dot{\varepsilon}$. Then the boundary conditions for $r = R$ are trivially satisfied.

Observing from (6.3) that $2\lambda\dot{a} + (\lambda + 2\mu)\dot{b} = 0$, the remaining boundary conditions are

$$(6.10)\qquad \left\{\begin{array}{l} Bc_n - 2\mu k_n b_n = 0 \\ \\ (\lambda + 2\mu)d_n - 2\mu k_n a_n = 0 \end{array}\right\} \text{ for } z = \pm h\,, \quad n = 0,1,2,\ldots\,.$$

For $n = 0$ the first equation of (6.10) is trivially satisfied and the second becomes $A_o = 0$. Hence $b_o' = 0$ and b_o = constant. The value of b_o determines a rigid displacement along the axis of the cylinder, so without loss of generality we may take $b_o = 0$.

For each integer $n \geq 1$, (6.10) gives us four linear homogeneous equations in A_n, B_n, C_n, and D_n. In order that the perturbed buckled solution differs from the perturbed simple solution, the determinant of coefficients of this linear system must be zero for some $n \geq 1$. This condition is

$$\overset{o}{\varepsilon} = \frac{2(\lambda+\mu)\left(\mp 1 - \dfrac{\sinh 2k_n h}{2k_n h}\right)}{\pm\lambda - (5\lambda + 4\mu)\dfrac{\sinh 2k_n h}{2k_n h}}$$

for some $n \geq 1$ where the proper sign is to be chosen. If the upper sign is chosen, it will be seen that the thinner the plate the greater the critical strain will be which is physically very strange. Also the critical strain corresponding to the lower sign is always less than that corresponding to the upper sign. For these reasons we expect the physically realistic choice to be the lower sign, although it is interesting that two solutions for $\overset{o}{\varepsilon}$ exist.

Now let ε_n be defined by

$$(6.11) \qquad \varepsilon_n = \frac{2(\lambda+\mu)\left(\dfrac{\sinh 2k_n h}{2k_n h} - 1\right)}{\lambda + (5\lambda + 4\mu)\dfrac{\sinh 2k_n h}{2k_n h}}, \qquad n = 1,2,3,\ldots .$$

Then $\overset{\circ}{\varepsilon}$ must be one of the numbers ε_n for buckling (i.e. for A_n, B_n, C_n, D_n not all zero), and the series in (6.4) have only one non-zero term since $\overset{\circ}{\varepsilon}$ can equal only one of the ε_n (note that $\varepsilon_n \neq \varepsilon_m$ for $n \neq m$). Hence the n-th mode of buckling is given by

$$(6.12) \qquad \begin{cases} u^1 = (1-\varepsilon)r + \delta a_n \alpha_n + O(\delta^2) \\ u^3 = \left(1 + \dfrac{2\lambda\varepsilon}{\lambda+2\mu}\right)z + \delta b_n \beta_n + O(\delta^2) \end{cases}$$

where the $\overset{\circ}{\varepsilon}$ appearing in a_n and b_n is replaced by ε_n and ε is an unknown function of δ having the property that $\varepsilon \rightarrow \varepsilon_n$ as $\delta \rightarrow 0$.

Let $D = \frac{2h}{R}$ (the undeformed thickness to radius ratio) and j_n be the n-th positive zero of J_1. Then (6.11) can be rewritten as

$$(6.11') \qquad \varepsilon_n = \frac{\dfrac{\sinh Dj_n}{Dj_n} - 1}{\sigma + (2+\sigma)\dfrac{\sinh Dj_n}{Dj_n}}$$

As $n \rightarrow \infty$, $\varepsilon_n \rightarrow \frac{1}{2+\sigma}$ so that all the modes of buckling start before the deformed radius is as small as $\frac{1+\sigma}{2+\sigma}R$.

Let T_n be the pressure per unit deformed area at which the n-th buckled mode can start. Then (see discussion of pressure on unbuckled cylinder)

$$(6.13) \qquad T_n = \frac{E\varepsilon_n}{(1-\varepsilon_n)(1-\sigma+2\sigma\varepsilon_n)} .$$

As $n \to \infty$, $T_n \to T_\infty = \dfrac{E(2+\sigma)}{(2-\sigma)(1+\sigma)^2}$ so that all modes of buckling start before the finite pressure T_∞ is reached.

As $D \to \infty$ with n held fixed, $\varepsilon_n \to \frac{1}{2+\sigma}$ and $T_n \to T_\infty$. Thus the intervals containing all the ε_n and T_n values shrink down to the limits of the sequences $\{\varepsilon_n\}$ and $\{T_n\}$ respectively as $D \to \infty$.

To compare our result with that of the thin plate theory we observe that

$$\varepsilon_n = \frac{D^2 j_n^2}{12(1+\sigma)} + O(D^4 j_n^4)$$

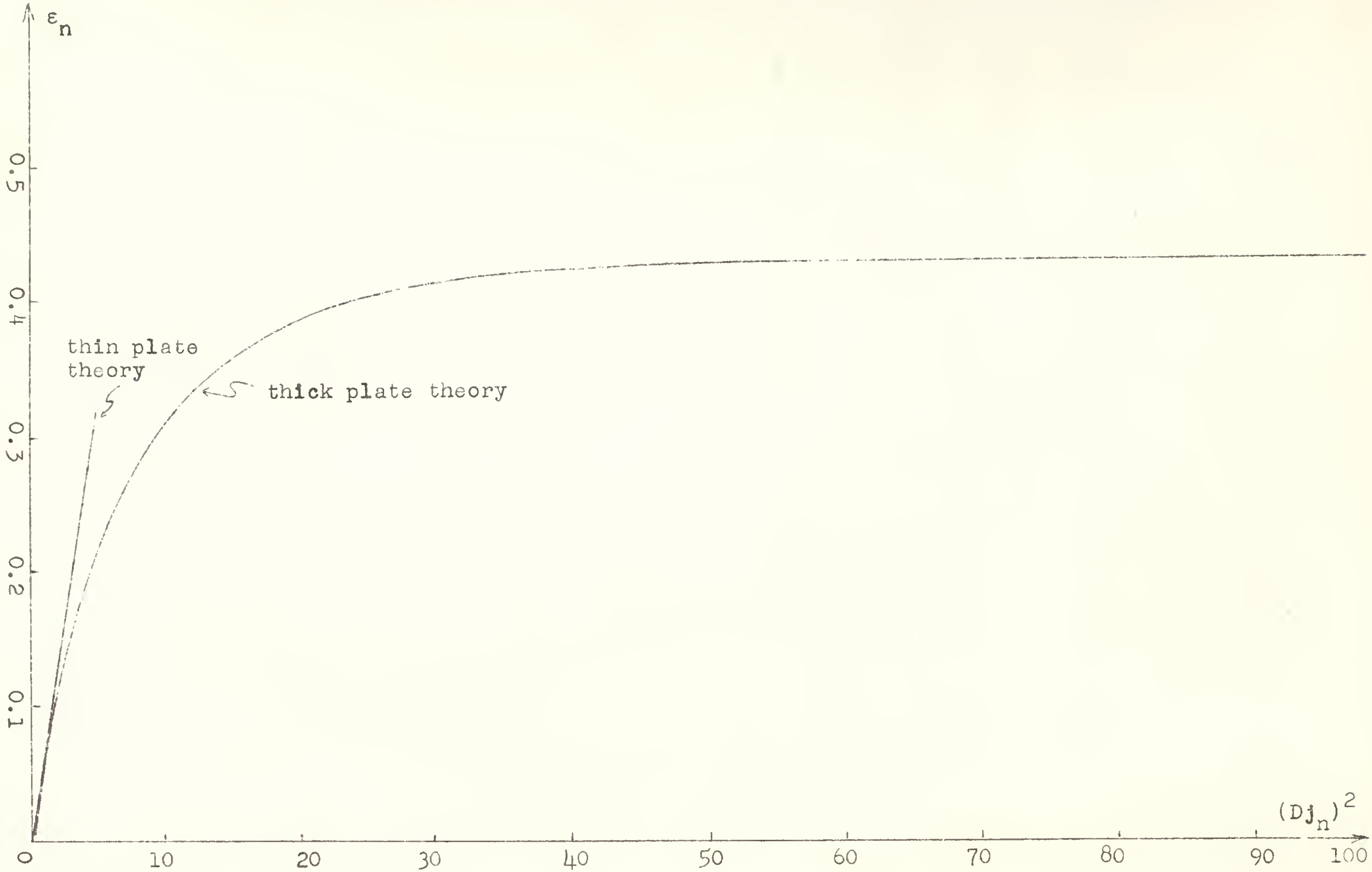
and

$$T_n = \frac{E}{1-\sigma}\varepsilon_n + O(\varepsilon_n^2) = \frac{ED^2 j_n^2}{12(1-\sigma^2)} + O(D^4 j_n^4) \ .$$

The force per unit deformed circumference is

$$2hbT_n = \frac{2Eh^3 j_n^2}{3(1-\sigma^2)R^2} + O(h^5 j_n^4) = F\left(\frac{j_n}{R}\right)^2 + O(h^5 j_n^4)$$

where $F = \dfrac{2Eh^3}{3(1-\sigma^2)}$ is the flexural rigidity of the plate. Thus to lowest order the critical pressure here agrees with that of the thin plate theory. In some respects, however, the two theories are quite different. Although the critical loads agree to lowest order, as $n \to \infty$ the thin plate critical loads $\to \infty$ whereas the thick plate critical loads approach a finite limit. This means that, no matter how thin the plate is, the critical loads for the two theories differ by arbitrarily large amounts for large enough n, i.e. for a buckling mode of higher order.



ε_n
0.5
0.4
0.3
0.2
0.1
0
thin plate theory
thick plate theory
$(Dj_n)^2$
10
20
30
40
50
60
70
80
90
100

Fig. 4

Figure 4 shows the results for this problem graphically for $\sigma = .3$. The critical strains ε_n are plotted versus $(Dj_n)^2$ where we recall that D is the undeformed thickness to radius ratio and j_n is the n-th positive zero of J_1. The graphs are shown for both the thick plate and thin plate theories.

7. Problem B

Cylindrical coordinates are used again. Since the axial displacement is taken to be zero, we have $\bar{x}_3 = u^3 = z$. In the unstrained cylinder we have $r_1 \leq r \leq r_2$ where r_1 and r_2 are the inner and outer radii. The height of the undeformed cylinder does not effect the results and is arbitrary.

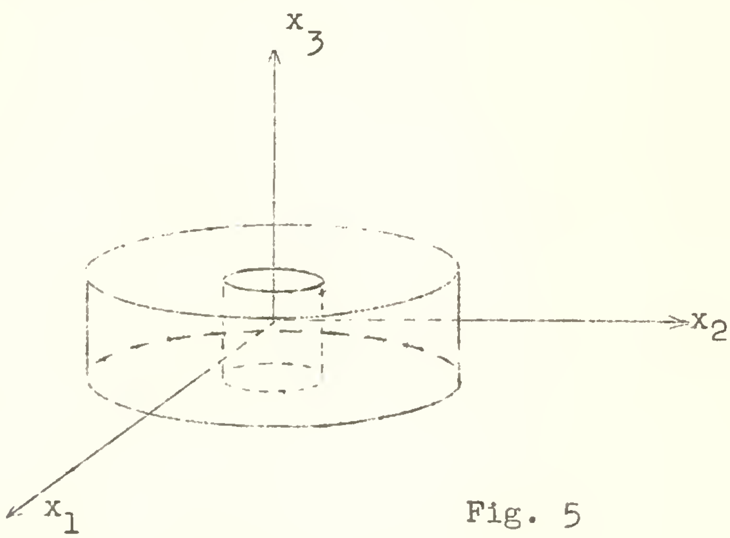


Fig. 5

The boundary conditions are

$$(7.1) \qquad \begin{cases} T^i = 0 \quad (i = 1,2,3) & \text{for } r = r_1 \\ T^i = -\bar{T}\bar{N}^i \quad (i = 1,2,3) & \text{for } r = r_2 \end{cases}$$

where T is the hydrostatic pressure per unit deformed area applied to the surface $r = r_2$ and $\overline{N}^i \vec{g}_i$ is the unit outer normal vector to the deformed outer boundary.

For $r = r_2$, $dS = r\,d\theta dz$, $N_1 = 1$, $N_2 = N_3 = 0$. Hence

$$T^i d\overline{S} = Q^{ij} N_j dS = rQ^{i1} d\theta dz \; .$$

Also

$$\overline{N}^i \overline{g}_i d\overline{S} = \frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial z}\, d\theta dz = u^j\big|_2 \vec{g}_j \times u^k\big|_3 \vec{g}_k \, d\theta dz$$

$$= \varepsilon_{ijk} P^j{}_2 \, P^k{}_3 \, \vec{g}^i \, d\theta dz \; .$$

Thus we also have

$$T^i d\overline{S} = -T\overline{N}^i d\overline{S} = -T\varepsilon^{ijk} P_{j2} P_{k3} \, d\theta dz \; ,$$

and hence

$$Q^{i1} = -\frac{T}{r}\, \varepsilon^{ijk} P_{j2} P_{k3} \; .$$

After similar considerations for $r = r_1$, the conditions (7.1) become

$$\text{(7.2)} \qquad \begin{cases} Q^{i1} = 0 \quad (i = 1,2,3) & \text{for } r = r_1 \\ Q^{i1} = -\dfrac{1}{r}\, T\varepsilon^{ijk} P_{j2} P_{k3} & \text{for } r = r_2 \end{cases} \; .$$

We look for a simple solution in which the displacements are entirely radial. Hence the deformed cylinder will be another hollow circular cylinder. Since $\vec{g}_1$ is directed along the radius and $\vec{g}_2$ and $\vec{g}_3$ are perpendicular to it, this means we want a simple solution of the form $u^1 = f(r)$, $u^2 = 0$, $u^3 = z$. Then from (5.3)

$$(P^i{}_j) = \begin{pmatrix} f' & 0 & 0 \\ 0 & \frac{1}{r}f & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P(^{ij}) = \begin{pmatrix} f' & 0 & 0 \\ 0 & \frac{1}{r^3}f & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Assuming $f > 0$ and $f' > 0$, then (P^{ij}) is symmetric and positive definite so (3.3) implies $C^{ij} = g^{ij}$. Also from (3.6), $s_1 = f' + \frac{1}{r}f - 2$. Substituting into (3.9), we obtain only one non-trivial equation, $(f' + \frac{1}{r} f)' = 0$. Hence $f = ar + \frac{b}{r}$ for some constants a and b.

We now obtain $s_1 = 2(a-1)$ and (from (3.7'))

$$Q^{11} = 2(\lambda+\mu)(a-1) - \frac{2\mu b}{r^2}, \qquad Q^{21} = Q^{31} = 0 .$$

Substituting into (7.2) we obtain

$$2(\lambda+\mu)(a-1) - \frac{2\mu b}{r_2^2} = -T\left(a + \frac{b}{r_2^2}\right) ,$$

$$2(\lambda+\mu)(a-1) - \frac{2\mu b}{r_1^2} = 0 .$$

Let $k = \left(\frac{r_1}{r_2}\right)^2$. Then the solution to the above equation is

$$(7.3) \qquad \begin{cases} 1-a = \dfrac{T}{2(\lambda+\mu)(1-k) + T(1 + \frac{\lambda+\mu}{\mu} k)} \\[2ex] b = \dfrac{\lambda+\mu}{\mu}(a-1)r_1^2 . \end{cases}$$

The a and b given by (7.3) are valid if T is small enough so that the deformed cylinder remains hollow. The inside radius of the deformed cylinder is

$$f(r_1) = ar_1 + \frac{b}{r_1} = \frac{(\lambda+\mu)2\mu - T)(1-k)r_1}{[\mu + (\lambda+\mu)k]T + 2\mu(\lambda+\mu)(1-k)} .$$

From this we see that the inside strained radius goes monotonically from r_1 to zero as T goes from zero to 2μ (notice that $0 < k < 1$). Hence we will look for buckled solutions which start at pressures T in the range $0 < T < 2\mu$. For T in this range $f > 0$ and $f' > 0$ as we assumed above.

We will look for buckled solutions which are symmetrical with respect to a radius. Hence we may as well restrict ourselves to solutions symmetrical with respect to the x_1-axis. These solutions are characterized by saying that u^1 is an even function of θ and u^2 is an odd function of θ. Hence we write

$$(7.4) \qquad \begin{cases} \dot{u}^1 = \dot{a}r + \frac{\dot{b}}{r} + \sum_{n=0}^{\infty} a_n(r)\cos n\theta \\ \dot{u}^2 = \sum_{n=0}^{\infty} b_n(r)\sin n\theta \quad \text{with} \quad b_o = 0 \\ \dot{u}^3 = 0 . \end{cases}$$

From (5.3) we obtain

$$\dot{p}^{11} = \dot{a} - \frac{\dot{b}}{r^2} + \sum a_n' \cos n\theta$$

$$\dot{p}^{12} = - \sum \left(\frac{n}{r^2} a_n + \frac{1}{r} b_n\right)\sin n\theta$$

$$\dot{p}^{21} = \sum \left(b_n' + \frac{1}{r} b_n\right)\sin n\theta$$

$$\dot{p}^{22} = \frac{\dot{a}}{r^2} + \frac{\dot{b}}{r^4} + \sum \left(\frac{1}{r^3} a_n + \frac{n}{r^2} b_n\right)\cos n\theta$$

$$\dot{p}^{13} = \dot{p}^{23} = \dot{p}^{31} = \dot{p}^{32} = \dot{p}^{33} = 0 .$$

From (4.2-4) we have

$$\dot{C}^{12} = -\dot{C}^{21} = \frac{1}{2\overset{\circ}{a}}(\dot{P}^{12} - \dot{P}^{21}) = \frac{-1}{2\overset{\circ}{a}} \sum (b_n' + \frac{2}{r} b_n + \frac{n}{r^2} a_n)\sin n\theta$$

(the other $\dot{C}^{ij}$'s are zero)

$$\dot{s}_1 = \dot{P}^i{}_i = 2\dot{a} + \sum (a_n' + \frac{1}{r} a_n + nb_n)\cos n\theta .$$

Let

$$(7.5) \qquad \begin{cases} c_n = a_n' + \frac{1}{r} a_n + nb_n \\ d_n = b_n' + \frac{2}{r} b_n + \frac{n}{r^2} a_n . \end{cases}$$

Then

$$\dot{C}^{12} = -\frac{1}{2\overset{\circ}{a}} \sum d_n \sin n\theta$$

$$\dot{s}_1 = 2\dot{a} + \sum c_n \cos n\theta$$

$$\dot{P}^{1j}\Big|_j = \sum (c_n' - nd_n)\cos n\theta$$

$$\dot{P}^{2j}\Big|_j = \sum (d_n' + \frac{1}{r} d_n - \frac{n}{r^2} c_n)\sin n\theta$$

$$\dot{C}^{1j}\Big|_j = -\frac{1}{2\overset{\circ}{a}} \sum nd_n \cos n\theta$$

$$\dot{C}^{2j}\Big|_j = \frac{1}{2\overset{\circ}{a}} \sum (d_n' + \frac{1}{r} d_n)\sin n\theta .$$

Substituting into (4.6), we see that the variables separate and the following differential equations are obtained.

$$(7.6) \qquad \begin{cases} (\lambda + 2\mu)c_n' - nAd_n = 0 \\ A(d_n' + \frac{1}{r} d_n) - (\lambda + 2\mu)\frac{n}{r^2} c_n = 0 \end{cases}$$

where $A = \lambda + 2\mu - \frac{\lambda+\mu}{\overset{\circ}{a}}$.

Since $b_o = 0$ by choice, then $d_o = 0$ from (7.5). From (7.6) $c_o' = 0$ and $c_o = A_o$ = constant.

For $n = 1,2,\ldots$ the solution of (7.6) is

$$c_n = A(A_n r^n + \frac{B_n}{r^n}) ,$$

$$d_n = (\lambda + 2\mu)(A_n r^{n-1} - \frac{B_n}{r^{n+1}})$$

where A_n and B_n are constants.

Next we solve the system (7.5). For $n = 0$ we have $a_o' + \frac{1}{r} a_o = A_o$ and $a_o = \frac{1}{2} A_o r + \frac{C_o}{r}$ where C_o = constant. For $n = 1$ the solution is

$$a_1 = \frac{1}{8} (3A-\lambda-2\mu)A_1 r^2 + \frac{1}{2} (A+\lambda+2\mu)B_1 \log r + \frac{C_1}{r^2} + D_1 ,$$

$$b_1 = \frac{1}{8}[-A+3(\lambda+2\mu)]A_1 r + \frac{1}{2}(A-\lambda-2\mu)\frac{B_1}{r} - \frac{1}{2}(A+\lambda+2\mu)\frac{B_1}{r} \log r + \frac{C_1}{r^3} - \frac{D_1}{r}$$

where C_1 and D_1 are additional constants.

For $n = 2,3,\ldots$ the solution to (7.5) is

$$a_n = - \frac{Bn + 2A}{4(n+1)} A_n r^{n+1} - \frac{Bn + 2A}{4(n-1)} \frac{B_n}{r^{n-1}} + C_n r^{n-1} + \frac{D_n}{r^{n+1}} ,$$

$$b_n = \frac{B(n+2) + 2A}{4(n+1)} A_n r^n - \frac{B(n-2) - 2A}{4(n-1)} \frac{B_n}{r^n} - C_n r^{n-2} + \frac{D_n}{r^{n+2}}$$

where $B = \frac{\lambda+\mu}{\breve{a}}$ and C_n and D_n are additional constants.

Next we turn to the boundary conditions. Since $(\mathring{P}^{ij})$ is diagonal, from (7.2) we obtain

$$\dot{Q}^{i1} = - \frac{1}{r} \dot{T}\varepsilon^{i23} \mathring{P}_{22}\mathring{P}_{33} - \frac{1}{r} \mathring{T}(\varepsilon^{ij3} \dot{P}_{j2}\mathring{P}_{33} + \varepsilon^{i2k} \mathring{P}_{22}\dot{P}_{k3})$$

for $r = r_2$.

Thus for $r = r_2$,

$$\dot{Q}^{11} = -\dot{T}(\overset{\circ}{a} + \frac{\overset{\circ}{b}}{r}) - \overset{\circ}{T}(\dot{a} + \frac{\dot{b}}{r^2}) - \overset{\circ}{T} \sum (nb_n + \frac{1}{r} a_n)\cos n\theta ,$$

$$\dot{Q}^{21} = -\frac{1}{r^2} \overset{\circ}{T} \sum (na_n + rb_n)\sin n\theta, \quad \text{and} \quad \dot{Q}^{31} = 0.$$

Also from (4.5)

$$\dot{Q}^{11} = 2(\lambda+\mu)\dot{a} - \frac{2\mu\dot{b}}{r^2} + \sum [(\lambda+2\mu)c_n - 2\mu(\frac{1}{r} a_n + nb_n)]\cos n\theta ,$$

$$\dot{Q}^{21} = \sum [Ad_n - \frac{2\mu}{r} (b_n + \frac{n}{r} a_n)]\sin n\theta, \quad \text{and} \quad \dot{Q}^{31} = 0.$$

Since

$$2(\lambda+\mu)(a-1) - \frac{2\mu b}{r_2^2} = -T(a + \frac{b}{r_2^2})$$

(from before (7.3)), we see that

$$2(\lambda+\mu)\dot{a} - \frac{2\mu\dot{b}}{r^2} = -\dot{T}(\overset{\circ}{a} + \frac{\overset{\circ}{b}}{r^2}) - \overset{\circ}{T}(\dot{a} + \frac{\dot{b}}{r^2})$$

when $r = r_2$. Thus the boundary conditions for $r = r_2$ become

$$(7.6) \quad \left.\begin{cases} (\lambda+2\mu)c_n + (\overset{\circ}{T} - 2\mu)(\frac{1}{r} a_n + nb_n) = 0 \\ \\ Ad_n + (\overset{\circ}{T} - 2\mu)(\frac{n}{r^2} a_n + \frac{1}{r} b_n) = 0 \end{cases}\right\} \begin{array}{l} \text{for } r = r_2 \\ \text{and } n = 0,1,2,\ldots . \end{array}$$

Similarly

$$(7.7) \quad \left.\begin{cases} (\lambda+2\mu)c_n - 2\mu(\frac{1}{r} a_n + nb_n) = 0 \\ \\ Ad_n - 2\mu(\frac{n}{r^2} a_n + \frac{1}{r} b_n) = 0 \end{cases}\right\} \text{for } r = r_1 \text{ and } n = 0,1,2,\ldots .$$

In applying the boundary conditions to our solution, it is convenient to express all constants depending on $\overset{\circ}{T}$ in terms of a single parameter. The most convenient such parameter seems to be

$$(7.8) \qquad S = \frac{(\lambda + 2\mu)\, \overset{\circ}{T}}{(\lambda+\mu)(1-k)(2\mu - \overset{\circ}{T})} \quad .$$

Then $S > 0$ since $k < 1$ and $\overset{\circ}{T} < 2\mu$. In terms of this parameter we have

$$\overset{\circ}{T} = \frac{2\mu S(\lambda+\mu)(1-k)}{\lambda+2\mu + S(\lambda+\mu)(1-k)}$$

$$\overset{\circ}{a} = \frac{(\lambda+\mu)S + \lambda + 2\mu}{(\lambda + 2\mu)(1 + S)}$$

$$A = \frac{\mu(\lambda + 2\mu)}{(\lambda+\mu)S + \lambda + 2\mu}$$

$$B = \frac{(\lambda+\mu)(\lambda + 2\mu)(1+S)}{(\lambda+\mu)S + \lambda + 2\mu} \quad .$$

For $n = 0$, (7.6) and (7.7) give $A_o = C_o = 0$. Thus we have $a_o = 0$ in addition to $b_o = 0$.

For $n = 1$, (7.6) and (7.7) give $A_1 = B_1 = C_1 = 0$. Thus $a_1 = D_1$ and $b_1 = -\frac{D_1}{r}$. The contribution which D_1 makes to the position vector in the deformed body is $D_1 \cos\theta\, \vec{g}_1 - \frac{D_1}{r} \sin\theta\, \vec{g}_2 = D_1(1,0,0)$. Since this is a rigid displacement along the x_1-axis, without loss of generality we may choose $D_1 = 0$ so that $a_1 = b_1 = 0$.

For each integer $n \geq 2$, (7.6) and (7.7) are four linear homogeneous equations in A_n, B_n, C_n, D_n. Setting the determinant of coefficients equal to zero and simplifying we obtain

$$[n^2(1-k)^2 + (1-2k)k\lambda_n]S^2 + 2[n^2(1-k)^2 + k^2\lambda_n]S + n^2(1-k)^2 - k\lambda_n = 0$$

where $\lambda_n = k^n + k^{-n} - 2$. This is the condition that a_n and b_n are not both identically zero. The parameter S was used because of the relatively simple form of the above equation.

The solution of this equation is

$$S = \frac{-k^2 - h_n \pm k\sqrt{(1-k)^2 + 4h_n}}{k(1-2k) + h_n} = \frac{h_n - k}{-k^2 - h_n \mp k\sqrt{(1-k)^2 + 4h_n}}$$

where $h_n = \frac{n^2}{\lambda_n}(1-k)^2$ and the proper sign is to be chosen.

Let $\eta = 1-k$. For a thin shell, $\eta \approx 0$ and

$$k^n = 1 - n\eta + \frac{n(n-1)}{2}\eta^2 - \frac{n(n-1)(n-2)}{6}\eta^3 + \frac{n(n-1)(n-2)(n-3)}{24}\eta^4 + O(\eta^5),$$

$$k^{-n} = 1 + n\eta + \frac{n(n+1)}{2}\eta^2 + \frac{n(n+1)(n+2)}{6}\eta^3 + \frac{n(n+1)(n+2)(n+3)}{24}\eta^4 + O(\eta^5),$$

$$\lambda_n = n^2\eta^2 + n^2\eta^3 + \frac{n^2(n^2+11)}{12}\eta^4 + O(\eta^5),$$

$$h_n = 1 - \eta - \frac{n^2-1}{12}\eta^2 + O(\eta^3),$$

$$h_n - k = \frac{n^2-1}{12}\eta^2 + O(\eta^3),$$

$$-k^2 - h_n = -2 + 3\eta + \frac{n^2-13}{12}\eta^2 + O(\eta^3),$$

$$\sqrt{(1-k)^2 + 4h_n} = 2 - \eta - \frac{n^2-1}{12}\eta^2 + O(\eta^3),$$

$$k\sqrt{(1-k)^2 + 4h_n} = 2 - 3\eta - \frac{n^2-13}{12}\eta^2 + O(\eta^3),$$

$$-k^2 - h_n + k\sqrt{(1-k)^2 + 4h_n} = O(\eta^3).$$

Thus from the second expression for S we see that if the plus sign is chosen we will have $S = O(\frac{1}{\eta})$ and $S \to \infty$ as $\eta \to 0$. Since we want $S \to 0$ as $\eta \to 0$ (so that the critical pressure goes to zero as the thickness of the cylinder goes to zero), we see that the minus sign is the proper one. Thus we define

$$(7.9) \qquad \left.\begin{cases} \lambda_n = k^n + k^{-n} - 2 \\ h_n = \dfrac{n^2}{\lambda_n}(1-k)^2 \\ S_n = \dfrac{k - h_n}{k^2 + h_n + k\sqrt{(1-k)^2 + 4h_n}} \\ T_n = \dfrac{2\mu(\lambda+\mu)(1-k)S_n}{\lambda + 2\mu + (\lambda+\mu)(1-k)S_n} \end{cases}\right\} \text{for } n = 2,3,4,\ldots\,.$$

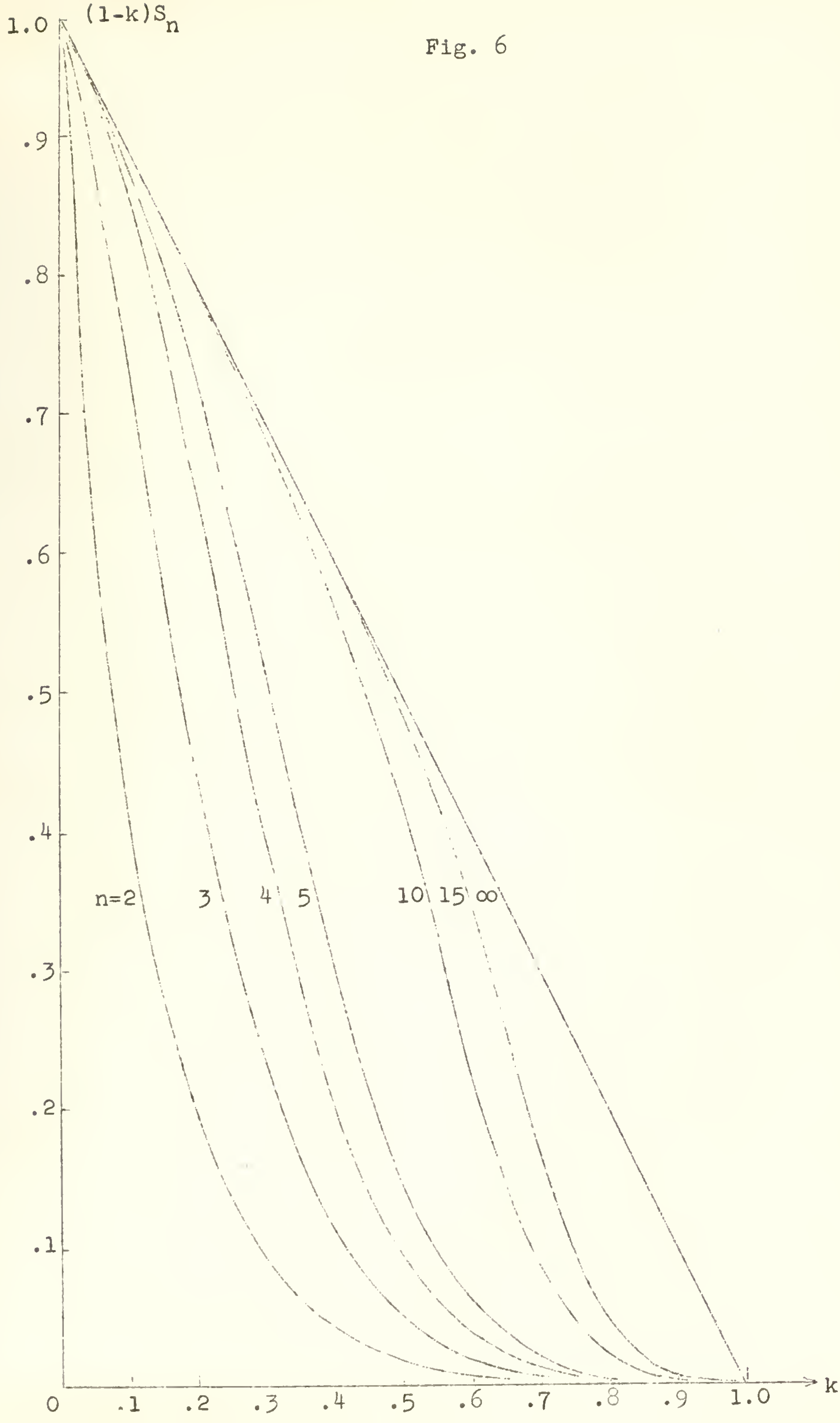
From (7.9) it can be shown that T_n increases monotonically with n so that there is at most one value of n for which a_n and b_n are not all identically zero. Hence the n-th buckled mode is

$$(7.10) \qquad \left.\begin{cases} u^1 = ar + \frac{b}{r} + \delta a_n \cos n\theta + O(\delta^2) \\ u^2 = \delta b_n \sin n\theta + O(\delta^2) \\ u^3 = z \end{cases}\right\} \text{for } n = 2,3,4,\ldots$$

where $\overset{\circ}{T}$ is replaced in a_n and b_n by T_n, the n-th critical pressure.

In Fig. 6 we have plotted $(1-k)S_n$ versus k for several values of n. For k = 0 the slope is -1 when n = 3,4,5,... and -9 when n = 2. For k = 1 the slope is zero for all finite n. From (7.9) we see that the graph of $T_n/2\mu$ versus k is a mild

Fig. 6



distortion of Fig. 6, but $T_n/2\mu$ depends on the elastic properties of the material whereas $(1-k)S_n$ does not.

T_2 agrees with Lubkin's result [3] (T_n for $n > 2$ is not derived in Lubkin's work), and to lowest order

$$T_n = \frac{\mu(\lambda+\mu)}{24(\lambda+2\mu)}(n^2-1)(1-k)^3 + O((1-k)^4)$$

$$= \frac{\mu(\lambda+\mu)}{3(\lambda+2\mu)}(n^2-1)\left(1-\frac{r_1}{r_2}\right)^3 + O\left(\left(1-\frac{r_1}{r_2}\right)^4\right)$$

which agrees with the results of thin plate methods.

As $n \to \infty$, $T_n \to T_\infty = \dfrac{2\mu(\lambda+\mu)(1-k)}{\lambda+2\mu+(\lambda+\mu)(1-k)} < 2\mu$. Thus all the T_n's are in the range for which the inside radius of the simple solution is positive. We notice that T_∞ depends on the relative dimensions of the hollow cylinder which is not the case in Problem A. However as $r_1 \to 0$ or $r_2 \to \infty$ (i.e. as $k \to 0$) with n fixed, each T_n converges to the same limit $\dfrac{2\mu(\lambda+\mu)}{2\lambda+3\mu}$. This behavior is similar to the behavior in Problem A as the thickness of the plate goes to infinity.

It is interesting to consider the case where the undeformed cylinder is solid. In order that there is no singularity at the axis of the cylinder, $f = ar$ and hence the stress is constant for the simple solution and B_n and $D_n = 0$ in the formulas for a_n, b_n, c_n, d_n. The only boundary conditions are (7.6), namely that we have hydrostatic pressure on the outside boundary. A calculation shows that (7.6) becomes

$$(\lambda+2\mu)AA_n r_2^n \pm (\dot{T}-2\mu)\left\{\left[\frac{n}{4}B + \frac{n-1}{2(n+1)}A\right]A_n r_2^n - (n-1)C_n r_2^{n-2}\right\} = 0\ .$$

Hence $A_n = C_n = 0$ (for $\overset{\circ}{T} \neq 2\mu$). Hence this procedure does not uncover a buckled solution unless the hydrostatic pressure is 2μ, and then all modes of buckling are possible, at least if only the first order perturbation problem is considered.

For the hollow cylinder, as the inside radius goes to zero, $T_n \to \frac{2\mu(\lambda+\mu)}{2\lambda+3\mu} < 2\mu$. Hence, as the inside radius goes to zero, the critical loads for the hollow cylinder do not approach that of the solid cylinder. Presumably this is due to the fact that in the limit the hollow cylinder has zero stress on its axis whereas the solid cylinder does not.

8. <u>Problem C</u>

Cylindrical coordinates are used again. The unstrained cylinder occuppies the region $r \leq R$, $0 \leq z \leq L$. The condition that the faces remain plane and horizontal can be written as $\bar{x}_3 = 0$ for $z = 0$, $\bar{x}_3 = L(1-\varepsilon)$ for $z = L$ where ε is a constant with $0 < \varepsilon < 1$. Since the faces remain horizontal and since $\vec{g}_1$ and $\vec{g}_2$ are horizontal vectors, the condition that no shear stress is developed on the faces can be written as $T^1 = T^2 = 0$ for $z = 0, L$. Thus the boundary conditions are

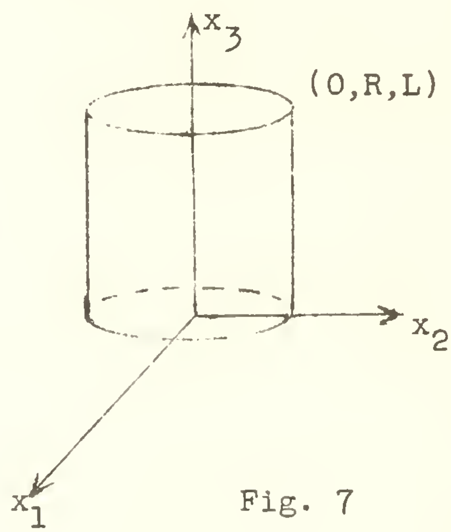


Fig. 7

$$(8.1) \quad \begin{cases} \bar{x}_3 = 0 & \text{for } z = 0 \\ \bar{x}_3 = L(1-\varepsilon) & \text{for } z = L \\ T^1 = T^2 = 0 & \text{for } z = 0, L \\ T^i = 0 \ (i = 1,2,3) & \text{for } r = R \ . \end{cases}$$

Using (3.2), the boundary conditions can be rewritten as

$$(8.2) \quad \begin{cases} u^3 = 0 & \text{for } z = 0 \\ u^3 = L(1-\varepsilon) & \text{for } z = L \\ Q^{13} = Q^{23} = 0 & \text{for } z = 0, L \\ Q^{i1} = 0 \ (i = 1,2,3) & \text{for } r = R \ . \end{cases}$$

We look for a simple solution such that $u^1 = ar$, $u^2 = 0$, $u^3 = bz$ where a and b are constants. Then

$$(P^i{}_j) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \text{ from (5.3) and } (P^{ij}) = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{a}{r^2} & 0 \\ 0 & 0 & b \end{pmatrix} .$$

Since (P^{ij}) is symmetric and positive definite (for $a,b > 0$), we have

$$C^{ij} = g^{ij} \qquad \text{(from (3.3))},$$

$$s_1 = 2a + b - 3 \qquad \text{(from (3.6)), and}$$

$$Q^{ij} = [\lambda(2a+b) - 3\lambda - 2\mu]g^{ij} + 2\mu P^{ij} \quad \text{(from (3.7'))} .$$

We now easily verify that the equations $Q^{ij}|_j = 0$ are satisfied and from (8.2) we have

$$(8.3)\qquad \begin{cases} a = 1 + \dfrac{\lambda\varepsilon}{2(\lambda+\mu)} = 1 + \sigma\varepsilon \\ b = 1 - \varepsilon \, . \end{cases}$$

Let T be the normal pressure per unit derformed area applied at z = 0,L to produce the simple solution. Then from (3.2) for z = L,

$$T = -T^3 = -Q^{33}\frac{dS}{d\bar{S}} = E\,\frac{\varepsilon}{a^2}\,.$$

Let P be the total load on an end. Then $P = \pi E\varepsilon R^2$.

As $\varepsilon \longrightarrow 1$, $T \longrightarrow \dfrac{E}{(1+\sigma)^2}$ and $P \longrightarrow \pi ER^2$. Hence the special strain energy density function used here has a consequence that a finite pressure and a finite total load would, as in previous cases, collapse the cylinder down to zero height and finite radius.

We consider buckled solutions such that the deformed axis of the cylinder remains in the x_1x_3-plane. We then expect the deformed cylinder to be consistent with

$$(8.4)\qquad \begin{cases} \dot{u}^1 = \dot{a}r + \sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} a_{mn}(r)\cos m\theta \cos k_n z \\ \dot{u}^2 = \sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} b_{mn}(r)\sin m\theta \cos k_n z \\ \dot{u}^3 = \dot{b}z + \sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} c_{mn}(r)\cos m\theta \sin k_n z \end{cases}$$

where $k_n = \frac{\pi n}{L}$ and $b_{on} = c_{mo} = 0$ without loss of generality.

From (5.3)

$$\dot{P}^{11} = \dot{a} + \sum\sum a'_{mn} \cos m\theta \cos k_n z$$

$$\dot{P}^{12} = -\sum\sum \left(\frac{m}{r^2} a_{mn} + \frac{1}{r} b_{mn}\right) \sin m\theta \cos k_n z$$

$$\dot{P}^{13} = -\sum\sum k_n a_{mn} \cos m\theta \sin k_n z$$

$$\dot{P}^{21} = \sum\sum \left(b'_{mn} + \frac{1}{r} b_{mn}\right) \sin m\theta \cos k_n z$$

$$\dot{P}^{22} = \frac{\dot{a}}{r^2} + \sum\sum \left(\frac{m}{r^2} b_{mn} + \frac{1}{r^3} a_{mn}\right) \cos m\theta \cos k_n z$$

$$\dot{P}^{23} = -\sum\sum k_n b_{mn} \sin m\theta \sin k_n z$$

$$\dot{P}^{31} = \sum\sum c'_{mn} \cos m\theta \sin k_n z$$

$$\dot{P}^{32} = -\sum\sum \frac{m}{r^2} c_{mn} \sin m\theta \sin k_n z$$

$$\dot{P}^{33} = \dot{b} + \sum\sum k_n c_{mn} \cos m\theta \cos k_n z \;.$$

From (4.2) we have $\dot{C}^{ij} = -\dot{C}^{ji}$ (remember $\overset{\circ}{C}{}^{ij} = g^{ij}$) and from (4.3)

$$\dot{C}^{12} = \frac{1}{2\overset{\circ}{a}}(\dot{P}^{12} - \dot{P}^{21})$$

$$= -\frac{1}{2\overset{\circ}{a}} \sum\sum \left(b'_{mn} + \frac{2}{r} b_{mn} + \frac{m}{r^2} a_{mn}\right) \sin m\theta \cos k_n z$$

$$\dot{C}^{13} = \frac{1}{\overset{\circ}{a} + \overset{\circ}{b}}(\dot{P}^{13} - \dot{P}^{31}) = \frac{-1}{\overset{\circ}{a} + \overset{\circ}{b}} \sum\sum (c'_{mn} + k_n a_{mn}) \cos m\theta \sin k_n z$$

$$\dot{C}^{23} = \frac{1}{\overset{\circ}{a} + \overset{\circ}{b}}(\dot{P}^{23} - \dot{P}^{32}) = \frac{1}{\overset{\circ}{a} + \overset{\circ}{b}} \sum\sum \left(\frac{m}{r^2} c_{mn} - k_n b_{mn}\right) \sin m\theta \sin k_n z$$

From (4.4)

$$\dot{s}_1 = \dot{P}^i{}_i = 2\dot{a} + \dot{b} + \sum\sum \left(a'_{mn} + \frac{1}{r} a_{mn} + m b_{mn} + k_n c_{mn}\right) \cos m\theta \cos k_n z$$

Let

$$(8.5)\quad \begin{cases} d_{mn} = a'_{mn} + \frac{1}{r} a_{mn} + m b_{mn} + k_n c_{mn} \\ e_{mn} = b'_{mn} + \frac{2}{r} b_{mn} + \frac{m}{r^2} a_{mn} . \end{cases}$$

Then

$$\dot{C}^{12} = -\frac{1}{2\overset{\circ}{a}} \sum\sum e_{mn} \sin m\theta \cos k_n z$$

$$\dot{s}_1 = 2\dot{a} + \dot{b} + \sum\sum d_{mn} \cos m\theta \cos k_n z .$$

From (5.4)

$$\dot{P}^{1j}\big|_j = \sum\sum (d'_{mn} - m e_{mn} - k_n c'_{mn} - k_n^2 a_{mn}) \cos m\theta \cos k_n z$$

$$\dot{P}^{2j}\big|_j = \sum\sum (e'_{mn} + \frac{1}{r} e_{mn} - \frac{m}{r^2} d_{mn} - k_n^2 b_{mn} + \frac{m k_n}{r^2} c_{mn}) \sin m\theta \cos k_n z$$

$$\dot{P}^{3j}\big|_j = \sum\sum [c''_{mn} + \frac{1}{r} c'_{mn} - (\frac{m^2}{r^2} + k_n^2) c_{mn}] \cos m\theta \sin k_n z$$

$$\dot{C}^{1j}\big|_j = -\sum\sum [\frac{m}{2\overset{\circ}{a}} e_{mn} + \frac{k_n}{\overset{\circ}{a} + \overset{\circ}{b}} (c'_{mn} + k_n a_{mn})] \cos m\theta \cos k_n z$$

$$\dot{C}^{2j}\big|_j = \sum\sum [\frac{1}{2\overset{\circ}{a}} (e'_{mn} + \frac{1}{r} e_{mn}) + \frac{k_n}{\overset{\circ}{a} + \overset{\circ}{b}} (\frac{m}{r^2} c_{mn} - k_n b_{mn})] \sin m\theta \cos k_n z$$

$$\dot{C}^{3j}\big|_j = \frac{1}{\overset{\circ}{a} + \overset{\circ}{b}} \sum\sum [c''_{mn} + \frac{1}{r} c'_{mn} - (\frac{m^2}{r^2} + k_n^2) c_{mn} + k_n d_{mn}] \cos m\theta \sin k_n z .$$

Substituting into (4.6) we see that the variables separate and the following ordinary differential equations are obtained:

$$(8.6)\quad \begin{cases} (\lambda+2\mu)d'_{mn} - \mu m e_{mn} - Bk_n(c'_{mn}+k_n a_{mn}) = 0 \\ \mu(e'_{mn}+\frac{1}{r}e_{mn}) - (\lambda+2\mu)\frac{m}{r^2}d_{mn} - Bk_n(k_n b_{mn} - \frac{m}{r^2}c_{mn}) = 0 \\ B[c''_{mn}+\frac{1}{r}c'_{mn} - (\frac{m^2}{r^2}+k_n^2)c_{mn}] - Ak_n d_{mn} = 0 \end{cases}$$

for $m,n = 0,1,2,\ldots$ where $A = \lambda + \frac{2\mu\mathring{a}}{\mathring{a}+\mathring{b}}$, $B = \frac{2\mu\mathring{b}}{\mathring{a}+\mathring{b}}$.

Eliminating c_{mn} from the first two equations of (8.6) we have

$$e''_{mn} + \frac{3}{r}e'_{mn} - (\frac{m^2-1}{r^2} + \frac{B}{\mu}k_n^2)e_{mn} = 0 .$$

Hence

$$e_{on} = 0 \quad \text{from (8.5) since } b_{on} = 0,$$

$$e_{mo} = (\lambda+2\mu)(A_{mo}r^{m-1} + \frac{D_{mo}}{r^{m+1}}) \qquad \text{for } m = 1,2,3,\ldots,$$

$$e_{mn} = \frac{1}{r}A_{mn}I_m(r_n^*) + \frac{1}{r}D_{mn}K_m(r_n^*) \qquad \text{for } m,n \neq 0$$

where $r_n^* = \sqrt{\frac{B}{\mu}}\,k_n r$ and I_m and K_m are the m-th order modified Bessel functions of the first and second kinds respectively.

Eliminating e_{mn} from the first two equations of (8.6) we obtain

$$(\lambda+2\mu)(d''_{mn} + \frac{1}{r}d'_{mn} - \frac{m^2}{r^2}d_{mn}) - Bk_n^2 d_{mn}$$
$$- Bk_n[c''_{mn} + \frac{1}{r}c'_{mn} - (\frac{m^2}{r^2}+k_n^2)c_{mn}] = 0 .$$

Adding this to k_n times the third equation of (8.6) (observe that $A+B = \lambda+2\mu$), we obtain

$$d''_{mn} + \frac{1}{r} d'_{mn} - (\frac{m^2}{r^2} + k_n^2) d_{mn} = 0 ,$$

and hence

$$d_{oo} = B_{oo} + E_{oo} \log r$$

$$d_{mo} = \mu(B_{mo} r^n + \frac{E_{mo}}{r^m}) \qquad \text{for } m = 1,2,3,\ldots$$

$$d_{mn} = B_{mn} I_m(\bar{r}_n) + E_{mn} K_m(\bar{r}_n) \quad \text{for } n \neq 0 , \quad m = 0,1,2,\ldots$$

where $\bar{r}_n = k_n r$.

The third equation of (8.6) now becomes

$$c''_{mn} + \frac{1}{r} c'_{mn} - (\frac{m^2}{r^2} + k_n^2) c_{mn} = \frac{Ak_n}{B} [B_{mn} I_m(\bar{r}_n) + E_{mn} K_m(\bar{r}_n)]$$

for $n \neq 0$ and

$$c''_{mo} + \frac{1}{r} c'_{mo} - \frac{m^2}{r^2} c_{mo} = 0 .$$

Thus

$$c_{mn} = \frac{A}{Bk_n} [\frac{1}{2} B_{mn} \bar{r}_n I'_m(\bar{r}_n) + C_{mn} I_m(\bar{r}_n) + \frac{1}{2} E_{mn} \bar{r}_n K'_m(\bar{r}_n) + F_{mn} K_m(\bar{r}_n)]$$

for $n \neq 0$, $m = 0,1,2,\ldots$.

From (8.6) we obtain $E_{oo} = 0$, $A_{mo} = B_{mo}$, $D_{mo} = -E_{mo}$ for $m \geq 1$.

To obtain a_{mn}, b_{mn} when $n \geq 1$ we solve the first two equations of (8.6) algebraically. To obtain a_{mo} and b_{mo} we solve (8.5). Listing the results we have

$$
(8.7)\quad \left\{
\begin{aligned}
a_{oo} &= \frac{1}{2} B_{oo} r + \frac{\alpha_o}{r} \\
a_{1o} &= -\frac{\lambda-\mu}{8} A_{1o} r^2 - \frac{\lambda+3\mu}{2} D_{1o} \log r - \frac{\alpha_1}{r^2} + \beta_1 \\
a_{mo} &= -\frac{\lambda m + \mu(m-2)}{4(m+1)} A_{mo} r^{m+1} + \frac{\lambda m + \mu(m+2)}{4(m-1)} \frac{D_{mo}}{r^{m-1}} \\
&\quad + \alpha_m r^{m-1} + \frac{\beta_m}{r^{m+1}} \quad \text{for } m \geq 2 \\
a_{mn} &= -\frac{\mu m}{Bk_n^2} \frac{1}{r} I_m(r_n^*) A_{mn} \\
&\quad + \left[\frac{\lambda+2\mu}{Bk_n} I_m'(\bar{r}_n) - \frac{A}{2Bk_n}\left(\frac{m^2}{\bar{r}_n} + \bar{r}_n\right) I_m(\bar{r}_n)\right] B_{mn} \\
&\quad - \frac{A}{Bk_n} I_m'(\bar{r}_n) C_{mn} - \frac{\mu m}{Bk_n^2} \frac{1}{r} K_m(r_n^*) D_{mn} \\
&\quad + \left[\frac{\lambda+2\mu}{Bk_n} K_m'(\bar{r}_n) - \frac{A}{2Bk_n}\left(\frac{m^2}{\bar{r}_n} + \bar{r}_n\right) K_m(\bar{r}_n)\right] E_{mn} \\
&\quad - \frac{A}{Bk_n} K_m'(\bar{r}_n) F_{mn} \quad \text{for } n \geq 1,\ m \geq 0 \\
b_{on} &= 0 \quad \text{for } n \geq 0 \text{ by choice} \\
b_{1o} &= \frac{3\lambda+5\mu}{8} A_{1o} r + \frac{\lambda+\mu}{2} \frac{D_{1o}}{r} + \frac{\lambda+3\mu}{2} \frac{D_{1o}}{r} \log r - \frac{\alpha_1}{r^3} - \frac{\beta_1}{r} \\
b_{mo} &= \frac{\lambda(m+2) + \mu(m+4)}{4(m+1)} A_{mo} r^m + \frac{\lambda(m-2) + \mu(m-4)}{4(m-1)} \frac{D_{mo}}{r^m} \\
&\quad - \alpha_m r^{m-2} + \frac{\beta_m}{r^{m+2}} \quad \text{for } m \geq 2 \\
b_{mn} &= \frac{1}{r_n^*} I_m'(r_n^*) A_{mn} + \left[\frac{A}{2B} \bar{r}_n I_m'(\bar{r}_n) - \frac{\lambda+2\mu}{B} I_m(\bar{r}_n)\right] \frac{m}{\bar{r}_n^2} B_{mn} \\
&\quad + \frac{A}{B} \frac{m}{\bar{r}_n^2} I_m(\bar{r}_n) C_{mn} + \frac{1}{r_n^*} K_m'(r_n^*) D_{mn} \\
&\quad + \left[\frac{A}{2B} \bar{r}_n K_m'(\bar{r}_n) - \frac{\lambda+2\mu}{B} K_m(\bar{r}_n)\right] \frac{m}{\bar{r}_n^2} E_{mn} \\
&\quad + \frac{A}{B} \frac{m}{\bar{r}_n^2} K_m(\bar{r}_n) F_{mn} \quad \text{for } m,n \geq 1
\end{aligned}
\right.
$$

$$
\begin{cases}
c_{mo} = 0 \quad \text{for } m \geq 0 \text{ by choice} \\[2ex]
c_{mn} = \dfrac{A}{Bk_n} \left[\dfrac{\bar{r}_n}{2} I'_m(\bar{r}_n)B_{mn} + I_m(\bar{r}_n)C_{mn} \right. \\[2ex]
\qquad \left. + \dfrac{\bar{r}_n}{2} K'_m(\bar{r}_n)E_{mn} + K_m(\bar{r}_n)F_{mn}\right] \quad \text{for } n \geq 1,\ m \geq 0 \\[2ex]
d_{oo} = B_{oo} \\[2ex]
d_{mo} = \mu\left(A_{mo}r^m - \dfrac{D_{mo}}{r^m}\right) \quad \text{for } m \geq 1 \\[2ex]
d_{mn} = I_m(\bar{r}_n)B_{mn} + K_m(\bar{r}_n)E_{mn} \quad \text{for } n \geq 1,\ m \geq 0 \\[2ex]
e_{on} = 0 \quad \text{for } n \geq 0 \\[2ex]
e_{mo} = (\lambda + 2\mu)\left(A_{mo}r^{m-1} + \dfrac{D_{mo}}{r^{m+1}}\right) \quad \text{for } m \geq 1 \\[2ex]
e_{mn} = \dfrac{1}{r} I_m(r_n^*)A_{mn} + \dfrac{1}{r} K_m(r_n^*)D_{mn} \quad \text{for } n,m \geq 1
\end{cases}
$$

In order that we have realistic behavior at the origin we want the components of $\dot{u}^1\vec{g}_1$, $\dot{u}^2\vec{g}_2$, and $\dot{u}^3\vec{g}_3$ to be bounded as $r \longrightarrow 0$. Hence we want a_{mn}, rb_{mn}, and c_{mn} to be bounded as $r \longrightarrow 0$. To accomplish this we choose the following to be zero: α_o, α_1, β_o, β_m $(m \geq 2)$, D_{mo} $(m \geq 1)$, $(A_{on}$, B_{on}, C_{on} for $n \geq 1)$, $(D_{mn}$, E_{mn}, F_{mn} for $m \geq 0$, $n \geq 1)$.

We immediately have $a_{on} = b_{on} = c_{on} = 0$ for $n \geq 1$.

Next we turn to the boundary conditions. First observe that from (4.5)

$$\dot{Q}^{13} = \frac{2\mu}{\overset{\circ}{a} + \overset{\circ}{b}} \sum\sum (\overset{\circ}{a} c'_{mn} - \overset{\circ}{b} k_n a_{mn}) \cos m\theta \sin k_n z$$

$$\dot{Q}^{23} = - \frac{2\mu}{\overset{\circ}{a} + \overset{\circ}{b}} \sum\sum (\overset{\circ}{b} k_n b_{mn} + \overset{\circ}{a} \frac{m}{r^2} c_{mn}) \sin m\theta \sin k_n z$$

$$\dot{Q}^{11} = 2(\lambda+\mu)\dot{a} + \lambda\dot{b} + \sum\sum (2\mu a'_{mn} + \lambda d_{mn}) \cos m\theta \cos k_n z$$

$$\dot{Q}^{21} = \mu \sum\sum (b'_{mn} - \frac{m}{r^2} a_{mn}) \sin m\theta \cos k_n z$$

$$\dot{Q}^{31} = \sum\sum [Bc'_{mn} - (A-\lambda) k_n a_{mn}] \cos m\theta \sin k_n z$$

and from (8.3) that $\dot{b} = -\dot{\varepsilon}$ and $2(\lambda+\mu)\dot{a} + \mu\dot{b} = 0$.

Then (8.2) gives

$$(8.8) \qquad \left\{ \begin{array}{l} 2\mu a'_{mn} + \lambda d_{mn} = 0 \\[2ex] b'_{mn} - \dfrac{m}{r^2} a_{mn} = 0 \\[2ex] Bc'_{mn} - (A-\lambda) k_n a_{mn} = 0 \end{array} \right\} \text{ for } r = R\ .$$

From (8.8) we obtain $B_{oo} = 0$, $A_{mo} = 0$ for $n \geq 1$, and $\alpha_m = 0$ for $m \geq 2$. Thus $a_{mo} = b_{mo} = 0$ for $m = 0,2,3,\ldots,$ $a_{1o} = \beta_1$, and $b_{1o} = - \frac{\beta_1}{r}$. We have now evaluated a_{mn}, b_{mn}, and c_{mn} when m or n = 0.

For $m,n > 0$ and $r = R$ equations (8.8) give

$$
(8.9)\quad \begin{cases}
-\frac{m}{r_n^*}[I_m'(r_n^*) - \frac{1}{r_n^*} I_m(r_n^*)]A_{mn} - \frac{A}{B} I_m''(\bar{r}_n)C_{mn} \\
\quad + \left\{-[\frac{\lambda+2\mu}{B} \frac{1}{\bar{r}_n} + \frac{A}{2B} (\frac{m^2}{\bar{r}_n} + \bar{r}_n)]I_m'(\bar{r}_n)\right. \\
\quad \left. + [(\frac{m^2}{\bar{r}_n^2}+1) \frac{\lambda+2\mu}{B} - \frac{A}{2B}(1-\frac{m^2}{\bar{r}_n^2}) + \frac{\lambda}{2\mu}]I_m(\bar{r}_n)\right\}B_{mn} = 0 \\
[-\frac{1}{r_n^*} I_m'(r_n^*) + (\frac{m^2}{r_n^{*2}} + \frac{1}{2})I_m(r_n^*)]A_{mn} \\
\quad + \frac{A}{B}\frac{m}{\bar{r}_n}[I_m'(\bar{r}_n) - \frac{1}{\bar{r}_n} I_m(\bar{r}_n)]C_{mn} \\
\quad + m\left\{\frac{A}{2B}[-\frac{1}{\bar{r}_n} I_m'(\bar{r}_n) + (\frac{m^2}{\bar{r}_n^2}+1)I_m(\bar{r}_n)]\right. \\
\quad \left. - \frac{\lambda+2\mu}{B\bar{r}_n}[I_m'(\bar{r}_n) - \frac{1}{\bar{r}_n} I_m(\bar{r}_n)]\right\}B_{mn} = 0 \\
-\mu \frac{\overset{\circ}{a}}{\overset{\circ}{b}} \frac{m}{\bar{r}_n} I_m(r_n^*)A_{mn} - \frac{2A\mu}{B} I_m'(\bar{r}_n)C_{mn} \\
\quad + \left\{(\lambda+2\mu)\frac{\overset{\circ}{a}}{\overset{\circ}{b}} I_m'(\bar{r}_n) - \frac{A\mu}{B}(\frac{m^2}{\bar{r}_n} + \bar{r}_n)I_m(\bar{r}_n)\right\}B_{mn} = 0
\end{cases}
$$

We look for a value ε_{mn} of $\overset{\circ}{\varepsilon}$ for which A_{mn}, B_{mn}, and C_{mn} are not all zero. The determinant of coefficients of (8.9) will be zero for $\overset{\circ}{\varepsilon}$ replaced by ε_{mn}, and it can be developed in a power series in R. Also ε_{mn} will be a function of R^2 (from (8.9)) and from physical considerations we have $\varepsilon_{mn} \longrightarrow 0$ as $R \longrightarrow 0$. Then $\varepsilon_{mn} = O(R^2)$, $\overset{\circ}{a} = 1 + O(R^2)$, $\overset{\circ}{b} = 1 + O(R^2)$, $A = \lambda + \mu + O(R^2)$, $B = \mu + O(R^2)$, and $\bar{r}_n = k_n R$ and $r_n^* = k_n R + O(R^3)$ (for $r = R$).

To lowest order in R we discover that the coefficients of the first two equations of (8.9) differ only in sign. Hence we replace the first equation by the sum of the first two obtaining

$$\left\{-\frac{m+1}{r_n^*} I_m'(r_n^*) + \left[\frac{m(m+1)}{r_n^{*2}} + \frac{1}{2}\right] I_m(r_n^*)\right\} A_{mn}$$

$$- \frac{A}{B}\left\{-\frac{m+1}{\bar{r}_n} I_m'(\bar{r}_n) + \left[\frac{m(m+1)}{\bar{r}_n^2} + 1\right] I_m(\bar{r}_n)\right\} C_{mn}$$

$$+ \left\{-\left[\frac{\lambda+2\mu}{B}\frac{m+1}{\bar{r}_n} + \frac{A}{2B}\left(\frac{m(m+1)}{\bar{r}_n} + \bar{r}_n\right)\right] I_m'(\bar{r}_n)\right.$$

$$\left. + \left[\frac{\lambda+2\mu}{B}\left(\frac{m(m+1)}{\bar{r}_n^2} + 1\right) + \frac{A}{2B}\left(\frac{m^2(m+1)}{\bar{r}_n^2} + m - 1\right) + \frac{\lambda}{2\mu}\right] I_m(\bar{r}_n)\right\} B_{mn} = 0$$

for $r = R$.

Retaining the lowest order non-zero term in the above coefficients we have

$$-\frac{\lambda+\mu}{2\mu} C_{mn} - \frac{\lambda+\mu}{4\mu}(m-2)B_{mn} + \ldots = 0.$$

Retaining the lowest order non-zero term in the coefficients of the last two equations of (8.9), we have

$$m(m-1)\left\{A_{mn} + \frac{\lambda+\mu}{\mu} C_{mn} + \left(\frac{\lambda+\mu}{2\mu} m - \frac{\lambda+2\mu}{\mu}\right) B_{mn}\right\} + \ldots = 0$$

$$m\left\{-\mu A_{mn} + [\lambda + 2\mu - (\lambda+\mu)m] B_{mn} - 2(\lambda+\mu) C_{mn}\right\} + \ldots = 0 .$$

Since $m \neq 0$, it follows that the determinant of coefficients of (8.9) is not zero unless $m = 1$. We conclude that for all $\overset{c}{\varepsilon}$ and $m,n > 0$ we have $A_{mn} = B_{mn} = C_{mn} = 0$ unless $m = 1$. Thus $a_{mn} = b_{mn} = c_{mn} = 0$ for $m,n > 0$ unless $m = 1$.

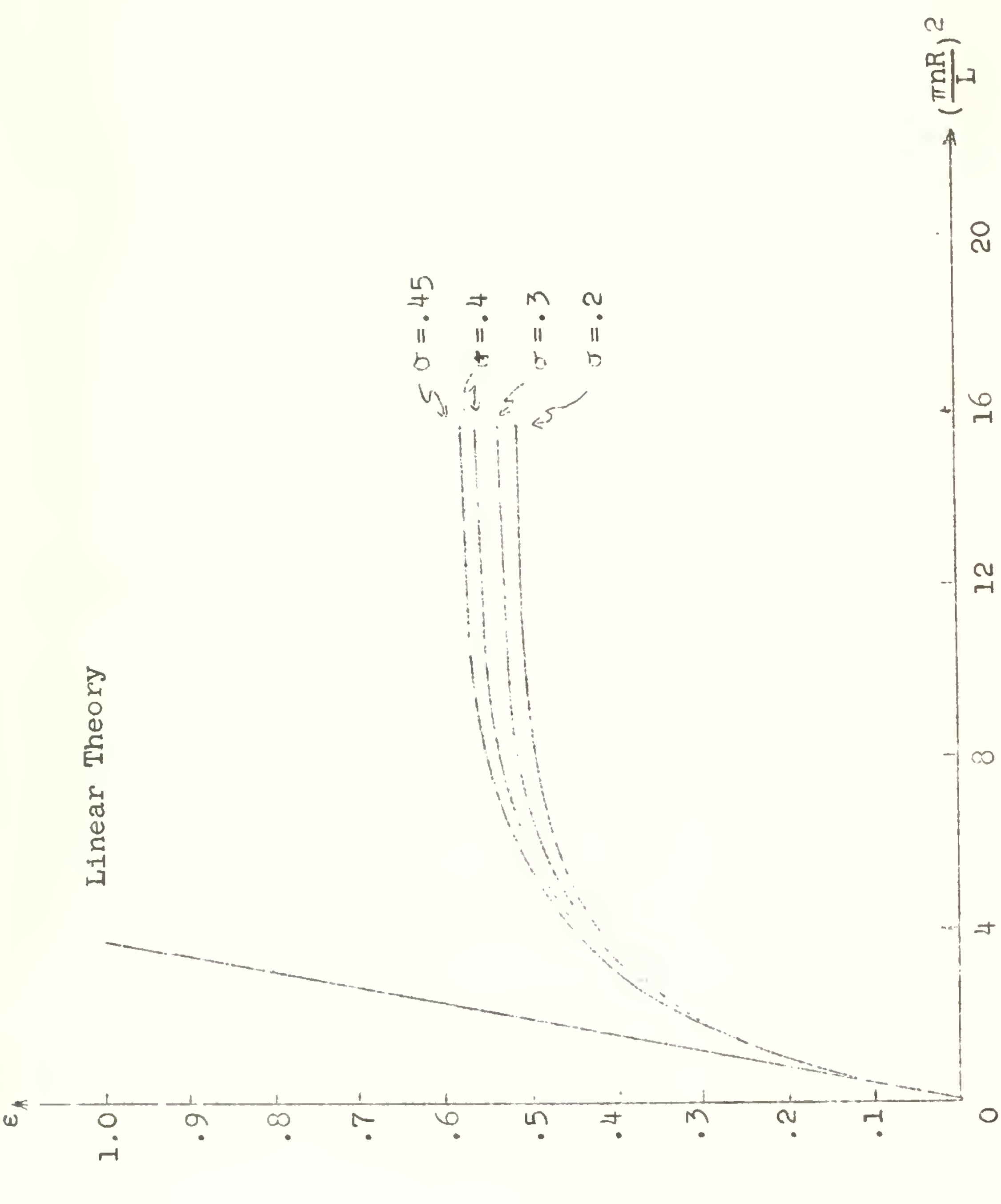
For $n \neq 0$, $m = 1$, and $r = R$, equations (8.9) become

$$
(8.10)\quad \begin{cases}
-\frac{1}{r_n^*}[I_1'(r_n^*)-\frac{1}{r_n^*}I_1(r_n^*)]A_{1n}-\frac{A}{B}I_1''(\bar{r}_n)C_{1n} \\
\quad +\left\{-(\frac{3A+2B}{2B}\frac{1}{\bar{r}_n}+\frac{A}{2B}\bar{r}_n)I_1'(\bar{r}_n)\right. \\
\quad \left.+(\frac{A+2B}{2B}+\frac{\lambda}{2\mu}+\frac{3A+2B}{2B}\frac{1}{\bar{r}_n^2})I_1(\bar{r}_n)\right\}B_{1n}=0 \\
-\frac{1}{r_n^*}I_1'(r_n^*)+(\frac{1}{r_n^{*2}}+\frac{1}{2})I_1(r_n^*)]A_{1n} \\
\quad +\frac{A}{B}\frac{1}{\bar{r}_n}[I_1'(\bar{r}_n)-\frac{1}{\bar{r}_n}I_1(\bar{r}_n)]C_{1n} \\
\quad +\left\{-\frac{3A+2B}{2B}\frac{1}{\bar{r}_n}[I_1'(\bar{r}_n)-\frac{1}{\bar{r}_n}I_1(\bar{r}_n)]+\frac{A}{2B}I_1(\bar{r}_n)\right\}B_{1n}=0 \\
-\frac{\overset{\circ}{a}}{\overset{\circ}{b}}\frac{1}{\bar{r}_n}I_1(r_n^*)A_{1n}+[\frac{\lambda+2\mu}{\mu}\frac{\overset{\circ}{a}}{\overset{\circ}{b}}I_1'(\bar{r}_n)-\frac{A}{B}(\frac{1}{\bar{r}_n}+\bar{r}_n)I_1(\bar{r}_n)]B_{1n} \\
\quad -\frac{2A}{B}I_1'(\bar{r}_n)C_{1n}=0\,.
\end{cases}
$$

Replacing $\overset{\circ}{\varepsilon}$ by ε_{1n} in equations (8.10) and setting the determinant of coefficients equal to zero, we obtain an equation for the ε_{1n}'s. Buckled solutions can occur only if $\overset{\circ}{\varepsilon}$ equals one of the ε_{1n}'s, and the n-th buckled mode has the form

$$
\left.\begin{aligned}
\dot{u}^1 &= \dot{a}r+(\beta_1+a_{1n}\cos k_n z)\cos\theta \\
\dot{u}^2 &= (-\frac{\beta_1}{r}+b_{1n}\cos k_n z)\sin\theta \\
\dot{u}^3 &= \dot{b}z+c_{1n}\cos\theta\sin k_n z
\end{aligned}\right\}\quad n=1,2,3,\ldots
$$

where β_1 is a constant. Since β_1 corresponds to a rigid displacement along the x_1-axis, we may choose $\beta_1 = 0$.



ε
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
4
8
12
16
20
$(\frac{\pi nR}{L})^2$
Linear Theory
σ = .45
σ = .4
σ = .3
σ = .2

Fig. 8

From (8.10) we see that ε_{1n} is a function of σ and $(\frac{\pi nR}{L})^2$. In Fig. 8 we have plotted $\varepsilon = \varepsilon_{1n}$ versus $(\frac{\pi nR}{L})^2$ for several values of σ. The graph for the linear theory is also shown. In the linear theory, the critical value of ε is independent of σ. Figure 8 shows that this is also nearly true in the non-linear theory when $\frac{\pi nR}{L}$ is small. It appears that ε approaches a finite limit as $\frac{\pi nR}{L} \longrightarrow \infty$ as was the case in Problem A.

The numerical procedure used to determine the points for Fig. 8 broke down for $(\frac{\pi nR}{L})^2$ somewhat larger than 16 since the procedure used for calculating I_1 and I_1' was valid only for $\frac{\pi nR}{L}$ small enough. However, there would be no essential difficulty in carrying the calculations further by other numerical methods.

9. Problem D

Much of the work carried out for Problem C can be used directly for Problem D. If we let r_1 and r_2 be the inner and outer radii of the undeformed hollow cylinder, the boundary conditions (8.1) still apply with the addition that the last boundary condition is valid for $r = r_1, r_2$. From these we obtain (8.2) where again the last condition is valid for $r = r_1, r_2$.

The simple solution of Problem C provides a simple solution for Problem D with a and b given by (8.3).

We look for a perturbed solution of the form (8.4). The differential equations are the same as those of Problem C so that the general solution is (8.7).

The boundary conditions (8.8) are valid for $r = r_1, r_2$. From them we obtain $a_{oo} = b_{oo} = 0$, $a_{1o} = \beta_1$, $b_{1o} = -\beta_1/r$, $a_{mo} = b_{mo} = 0$ for $m = 2,3,4,\ldots$ with one exception. For a certain unique value of r_1/r_2, a_{3o} and b_{3o} do not have to be zero. However this solution might not correspond to a buckled solution since in the first order perturbation problem the possibility of its occurrence depends only on the geometry of the undeformed cylinder rather than on the loading. We ignore this possible buckled solution rather than go to higher order perturbation problems.

Since β_1 corresponds to a rigid displacement along the x_1-axis, we take $\beta_1 = 0$.

For $m = 0,1,2,\ldots$, $n = 1,2,3,\ldots$, the boundary conditions become linear equations in A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, F_{mn}. The terms in A_{mn}, B_{mn}, C_{mn} are given by equations (8.9). The coefficients of D_{mn}, E_{mn}, F_{mn} can be obtained by replacing I_m and its derivatives by K_m and its derivatives in the coefficients of A_{mn}, B_{mn}, and C_{mn} respectively. The equations are to be valid for $r = r_1, r_2$. This gives four equations in four unknowns when $m = 0$ and six equations in six unknowns when $m \neq 0$. As previously we let ε_{mn} be the value of $\overset{o}{\varepsilon}$ which makes the determinant of coefficients equal to zero. The ε_{mn}'s are then the critical strains at which buckling can start.

From (8.9) we see that ε_{mn} is a function of σ, m, $\frac{\pi n r_1}{L}$, and $\frac{\pi n r_2}{L}$. Figures 9-13 give an indication of the dependence of ε_{mn} on some of these parameters. Each figure shows the graph of ε_{mn} versus $100\,\frac{r_1}{r_2}$ for $\sigma = .3$, one value of $\frac{\pi n r_2}{L}$, and $m = 0,1,2,3,4$.

If, for example, we consider a hollow cylinder with $\frac{\pi r_2}{L} = 1$, then Figures 9-13 give ε_{mn} for $n = 1,2,3,4,5$, $m = 0,1,2,3,4$, and all inner radii. In this example we see that $m = 1$, $n = 1$ is the first buckled mode which can occur when the inner radius is small enough. Also as the inner radius goes to zero, the lowest critical strain ε_{11} approaches .191 which is the critical strain for the solid cylinder (see Fig. 8). Hence the hollow cylinder behaves like a solid cylinder when the inside radius is small. As the inside radius becomes large, we see that the lowest value of $\varepsilon_{mn} \rightarrow 0$, so the thinner the cylindrical shell, the sooner it will buckle. The graphs show that as the cylindrical shell becomes thinner, the value of m for the first mode becomes larger since ε_{mn} is monotone decreasing in m for fixed n and $r_1/r_2 = 1$, and the value of n for the first mode also becomes larger since the graphs become flatter at $r_1/r_2 = 1$ as n increases. Hence, as we would expect, the first buckled mode is more crinkled as the cylindrical shell becomes thinner.

We do not mean to imply that the above conclusions hold for all hollow cylinders. For example, if $\frac{\pi r_2}{L} = 5$, Fig. 13 shows that $\varepsilon_{o1} < \varepsilon_{11}$ for inner radii small enough (although rather large). This gives rise to the possibility that hollow cylinders of large outside radius and small inside radius do not buckle like a solid cylinder. Of course we expect such cylinders to fail by crushing before they buckle anyhow.

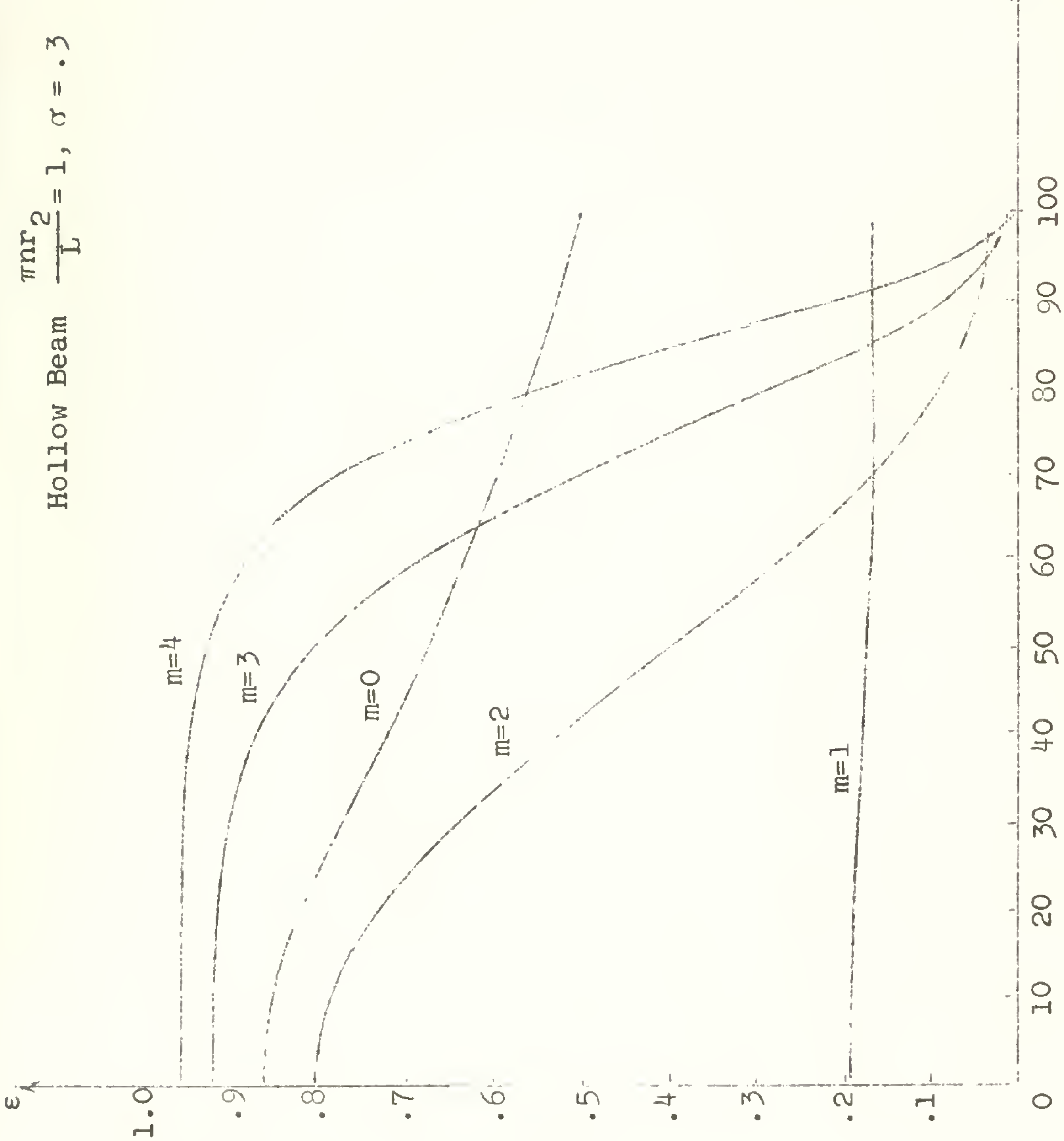
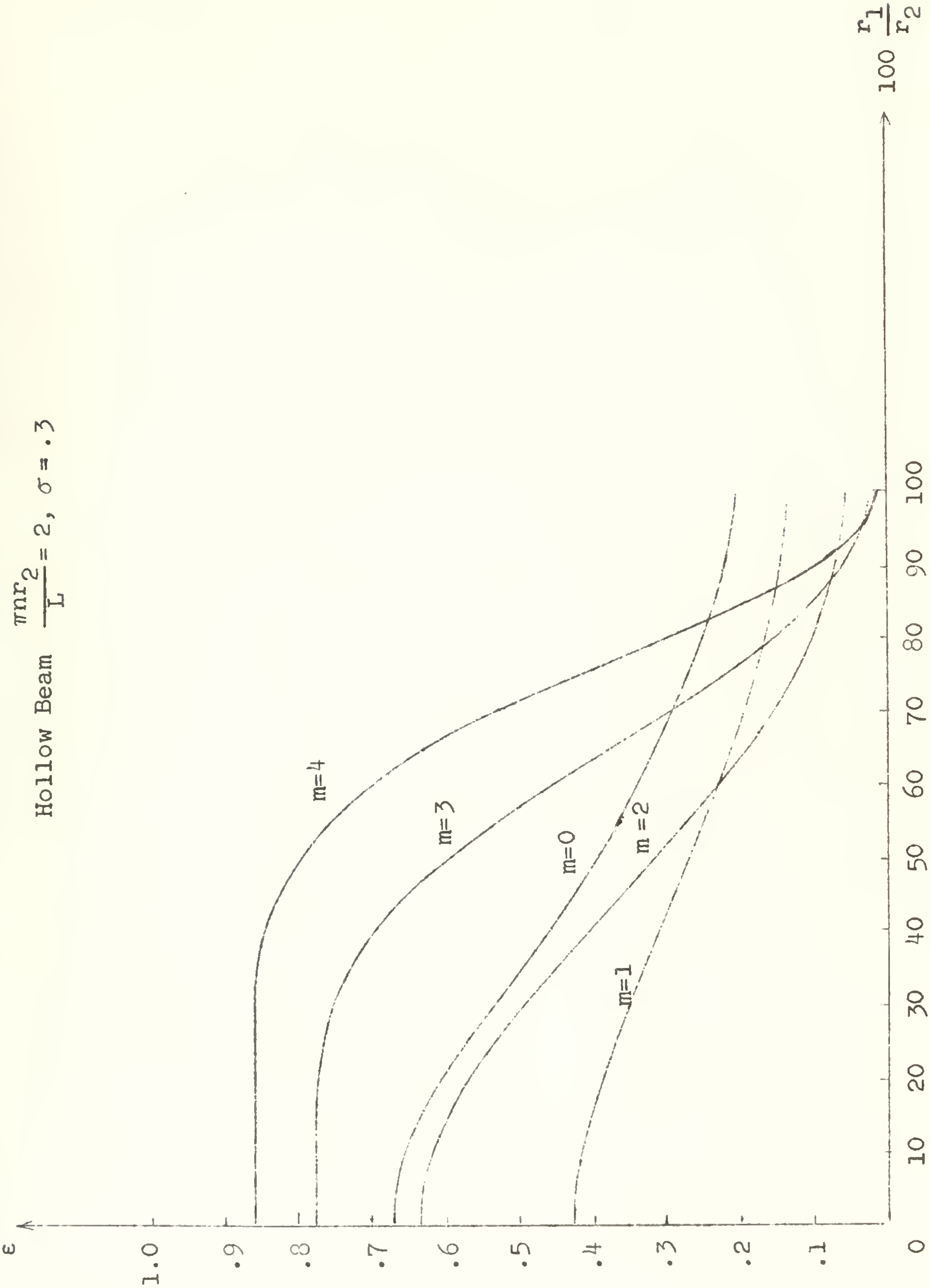

Hollow Beam $\frac{\pi n r_2}{L} = 1$, $\sigma = .3$
m=4
m=3
m=0
m=2
m=1
ε
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100
$100 \frac{r_1}{r_2}$

Fig. 9



Hollow Beam $\frac{\pi n r_2}{L} = 2$, $\sigma = .3$
ε
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100
$100 \frac{r_1}{r_2}$
m=4
m=3
m=0
m=2
m=1

Fig. 10

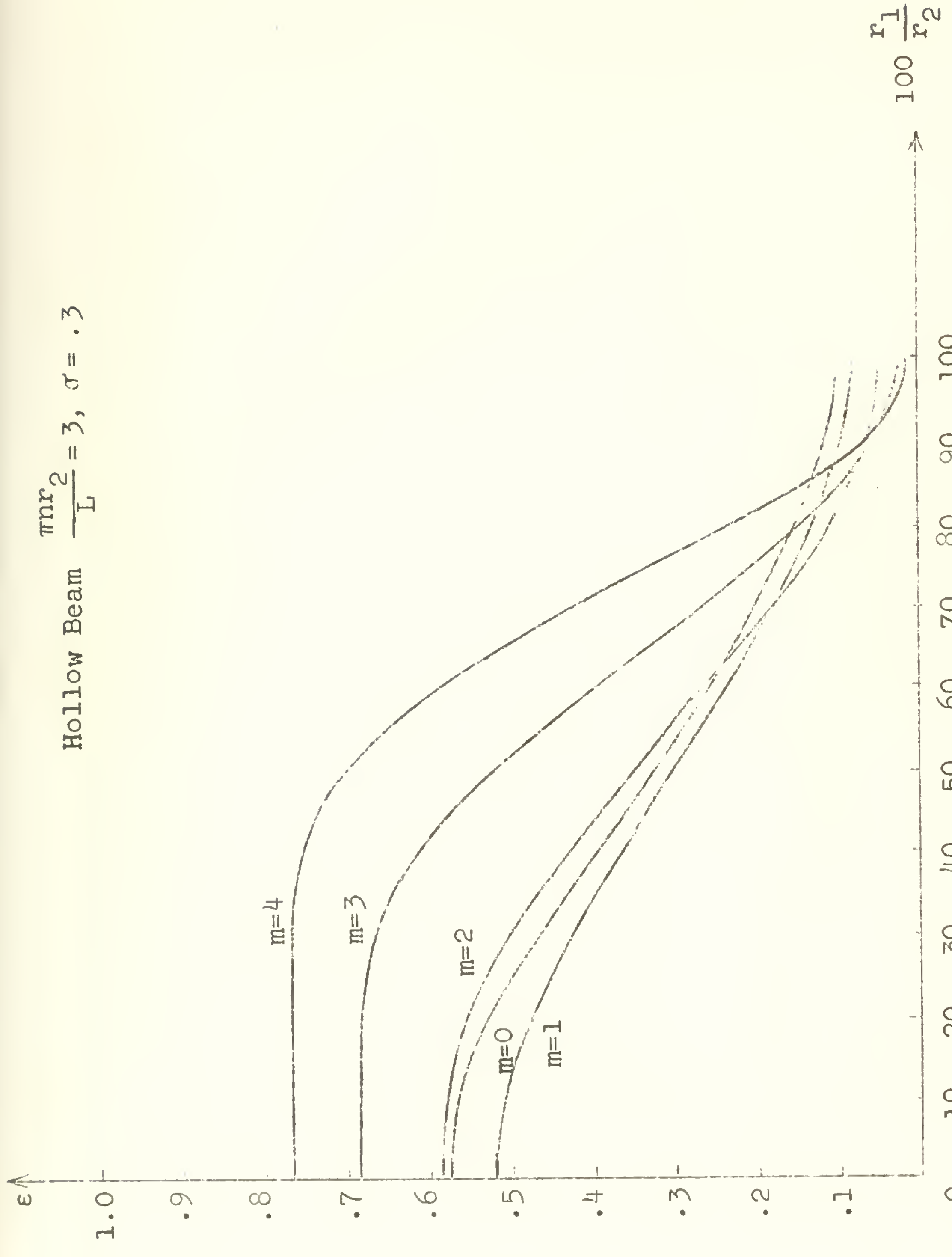
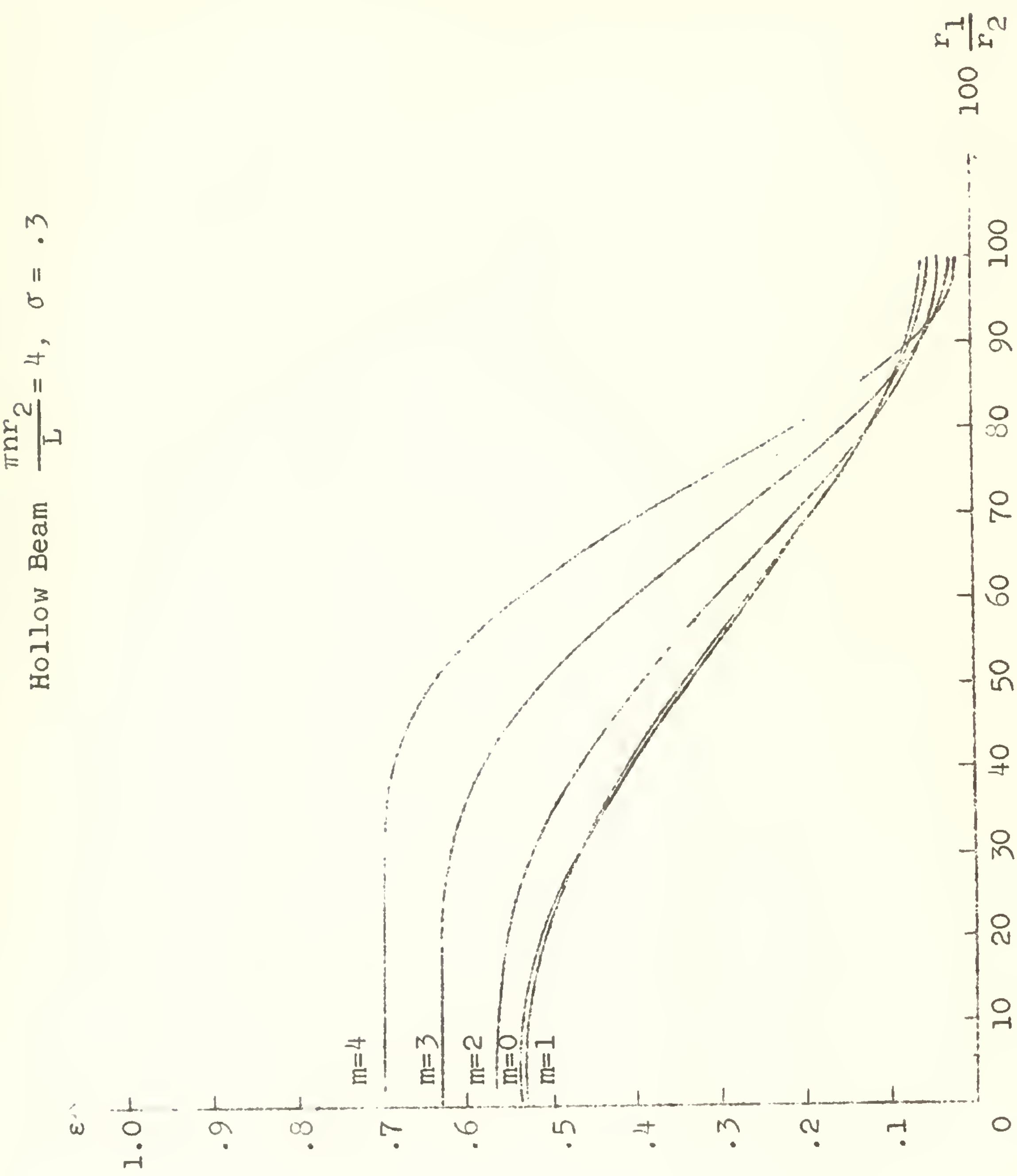
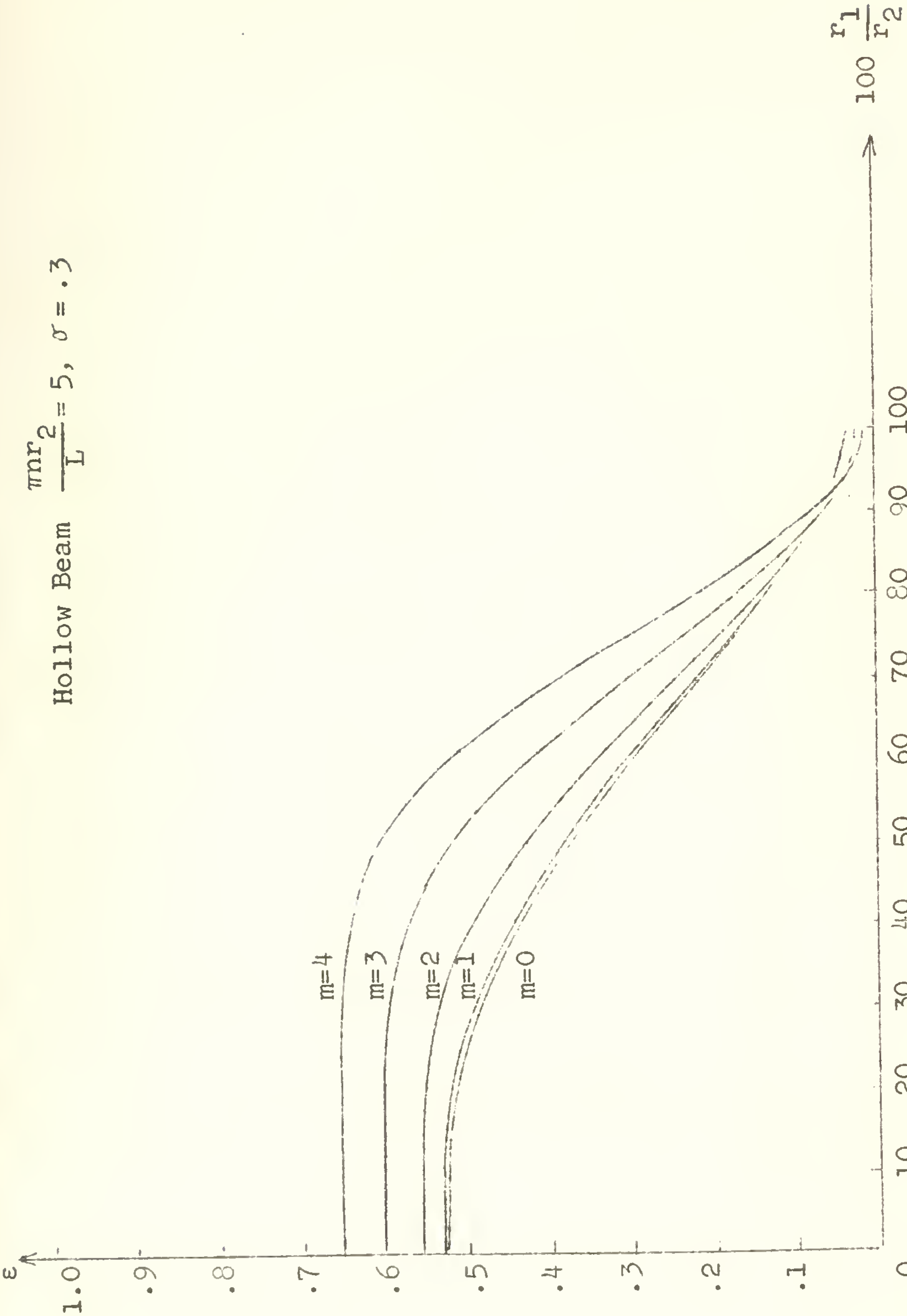

Hollow Beam $\frac{\pi n r_2}{L} = 3$, $\sigma = .3$
ε
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100
$100 \frac{r_1}{r_2}$
m=4
m=3
m=2
m=0
m=1

Fig. 11



Hollow Beam
$\frac{\pi n r_2}{L} = 4, \ \sigma = .3$
ε
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
m=4
m=3
m=2
m=0
m=1
10
20
30
40
50
60
70
80
90
100
$100 \frac{r_1}{r_2}$

Fig. 12



Hollow Beam $\frac{\pi n r_2}{L} = 5$, $\sigma = .3$
ε
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10
20
30
40
50
60
70
80
90
100
$100\frac{r_1}{r_2}$
m=4
m=3
m=2
m=1
m=0

Fig. 13

10. Problem E

Spherical coordinates are used (see Section 5). The unstrained region is $r_1 \leq r \leq r_2$. The boundary conditions are:

$$(10.1) \qquad \begin{cases} T^i = -T\overline{N}^i & \text{for } r = r_2 \\ T^i = 0 & \text{for } r = r_1 \end{cases}$$

where T is the hydrostatic pressure per unit deformed area.

For $r = r_2$, $dS = r^2 \sin\theta \, d\theta d\phi$, $N_1 = 1$, $N_2 = N_3 = 0$. Thus

$$T^i d\overline{S} = Q^{ij} N_j dS = r^2 \sin\theta \, Q^{i1} \, d\theta d\phi \, .$$

Also

$$\overline{N}^i \vec{g}_i d\overline{S} = \frac{\partial \vec{\overline{x}}}{\partial \theta} \quad \frac{\partial \vec{\overline{x}}}{\partial \phi} d\theta d\phi = u^j\big|_2 \vec{g}_j \times u^k\big|_3 \vec{g}_k \, d\theta d\phi$$

$$= \varepsilon_{ijk} P^j{}_2 \, p^k{}_3 \, \vec{g}^1 \, d\theta d\phi \, ,$$

Thus we also have

$$T^i d\overline{S} = -T\overline{N}^i d\overline{S} = -T\varepsilon^{ijk} P_{j2} P_{k3} \, d\theta d\phi$$

and hence

$$Q^{i1} = -\frac{T}{r^2 \sin\theta} \varepsilon^{ijk} P_{j2} P_{k3} \, .$$

After similar considerations on $r = r_1$, the conditions (10.1) become

$$(10.2) \qquad \begin{cases} Q^{i1} = -\dfrac{T}{r^2 \sin\theta} \varepsilon^{ijk} P_{j2} P_{k3} & \text{for } r = r_2 \\ Q^{i1} = 0 & \text{for } r = r_1 \end{cases}$$

We look for a simple solution having spherical symmetry. That is, a solution of the form $u^1 = f(r)$, $u^2 = u^3 = 0$. Then

$$(P^i{}_j) = \begin{pmatrix} f' & 0 & 0 \\ 0 & \frac{1}{r}f & 0 \\ 0 & 0 & \frac{1}{r}f \end{pmatrix}, \qquad (P^{ij}) = \begin{pmatrix} f' & 0 & 0 \\ 0 & \frac{1}{r^3}f & 0 \\ 0 & 0 & \frac{f}{r^3 \sin^2\theta} \end{pmatrix},$$

$C_{ij} = g_{ij}$, $s_1 = f' + \frac{2}{r}f - 3$, $P^{1j}|_j = \frac{\partial}{\partial r}(f' + \frac{2}{r}f)$, $P^{2j}|_j = P^{3j}|_j = 0$.

Hence (3.9) reduces to only one non-trivial equation, namely $\frac{\partial}{\partial r}(f' + \frac{2}{r}f) = 0$.

Thus $f = ar + \frac{b}{r^2}$ where a and b are constants. It follows that $s_1 = 3(a-1)$, $Q^{21} = Q^{31} = 0$, $Q^{11} = (3\lambda + 2\mu)(a-1) - \frac{4\mu b}{r^3}$.

The non-trivial boundary conditions (10.2) are $Q^{11} = -T(\frac{f}{r})^2$ for $r = r_2$ and $Q^{11} = 0$ for $r = r_1$. These become

$$(10.3) \qquad \begin{cases} (3\lambda + 2\mu)(a-1) - \dfrac{4\mu b}{r_2^3} = -\left(a + \dfrac{b}{r_2^3}\right)^2 T \\[2ex] (3\lambda + 2\mu)(a-1) - \dfrac{4\mu b}{r_1^2} = 0 \end{cases}$$

from which we obtain

$$(10.4) \qquad \begin{cases} b = \dfrac{r_1^3}{4\mu}(3\lambda + 2\mu)(a-1) \\[2ex] T = \dfrac{(3\lambda + 2\mu)(1-a)(1-k)}{[a + \frac{3\lambda + 2\mu}{4\mu}k(a-1)]^2} \end{cases}$$

where $k = \left(\frac{r_1}{r_2}\right)^3$ so that $0 < k < 1$.

The inside radius of the strained sphere is

$$f(r_1) = ar_1 + \frac{b}{r_1^2} = ar_1 + \frac{r_1}{4\mu}(3\lambda + 2\mu)(a-1) .$$

Since we want this to be non-negative, we have $1-a \leq \frac{4\mu}{3(\lambda+2\mu)}$. Since also $T \geq 0$, we have $1-a \geq 0$ from (10.4). Therefore we consider the range

$$0 \leq 1-a \leq \frac{4\mu}{3(\lambda+2\mu)} . \tag{10.5}$$

For the above range of a, T decreases monotonically from $\frac{12\mu(\lambda+2\mu)}{(1-k)(3\lambda+2\mu)}$ to zero as a increases monotonically. Thus as the hydrostatic pressure increases monotonically from zero to $\frac{12\mu(\lambda+2\mu)}{(1-k)(3\lambda+2\mu)}$, the inside strained radius decreases monotonically from r_1 to zero. Consequently buckled solutions which start at pressure T in this range are the only ones considered.

We will look for buckled solutions which are rotationally symmetric around the x_3-axis, i.e. for solutions of the form $u^1 = u^1(r,\theta)$, $u^2 = u^2(r,\theta)$, $u^3 = 0$.

We write

$$\begin{cases} \dot{u}^1 = \dot{f} + \sum a_n(r)\alpha_n(\theta) \\ \dot{u}^2 = \sum b_n(r)\beta_n(\theta) \end{cases} \tag{10.6}$$

and attempt to choose α_n and β_n so that we have separation of variables in (4.6). Then

$$\dot{P}^{11} = \dot{P}^{1}{}_{1} = \dot{f}' + \sum a_n'\alpha_n$$

$$\dot{P}^{12} = \frac{1}{r^2}\dot{P}^{1}{}_{2} = \frac{1}{r^2}\sum (a_n\alpha_n' - rb_n\beta_n)$$

$$\dot{P}^{13} = 0$$

$$\dot{P}^{21} = \dot{P}^{2}{}_{1} = \sum (b_n' + \frac{1}{r} b_n)\beta_n$$

$$\dot{P}^{22} = \frac{1}{r^2}\dot{P}^{2}{}_{2} = \frac{1}{r^2}[\frac{1}{r}\dot{f} + \sum (b_n\beta_n' + \frac{1}{r} a_n\alpha_n)]$$

$$\dot{P}^{23} = 0$$

$$\dot{P}^{31} = 0$$

$$\dot{P}^{32} = 0$$

$$\dot{P}^{33} = \frac{1}{r^2\sin^2\theta}\dot{P}^{3}{}_{3} = \frac{1}{r^2\sin^2\theta}[\frac{1}{r}\dot{f} + \sum (\frac{1}{r} a_n\alpha_n + b_n\beta_n \cot\theta)]$$

$$\dot{P}^{1j}\Big|_j = \sum [(a_n'' + \frac{2}{r} a_n' - \frac{2}{r^2} a_n)\alpha_n$$

$$+ \frac{1}{r^2} a_n(\alpha_n'' + \alpha_n' \cot\theta) - \frac{2}{r} b_n(\beta_n' + \beta_n \cot\theta)]$$

$$\dot{P}^{2j}\Big|_j = \sum [(b_n'' + \frac{4}{r} b_n' + \frac{2}{r^2} b_n)\beta_n$$

$$+ \frac{1}{r^2} b_n(\beta_n'' + \beta_n' \cot\theta - \frac{\beta_n}{\sin^2\theta}) + \frac{2}{r^3} a_n\alpha_n']$$

$$\dot{P}^{3j}\Big|_j = 0 .$$

To have separation of variables in the $\dot{P}^{ij}\big|_j$ themselves, it is sufficient to have $\alpha_n'' + \alpha_n' \cot\theta$ and $\beta_n' + \beta_n \cot\theta$ proportional to α_n and also to have $\beta_n'' + \beta_n' \cot\theta - \frac{\beta_n}{\sin^2\theta}$ and α_n' proportional to β_n. It is also to be hoped that a choice of α_n and β_n which accomplishes this will lead to a separation of

variables in (4.6). These proportionality conditions are met if we choose

$$(10.7)\quad \left.\begin{cases} \alpha_n(\theta) = P_n(\cos\theta) \\ \beta_n(\theta) = \alpha_n'(\theta) = -\sin\theta\, P_n'(\cos\theta) \end{cases}\right\} \text{for } n = 0,1,2,\ldots$$

where $P_n(x)$ is the Legendre polynomial defined by $P_o(x) = 1$, $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ for $n = 1,2,3,\ldots$. Then $\dot{u}^2$ will have a simple zero for $\theta = 0, \pi$, but this is desirable since we expect the solution to have that property.

The differential equation, $(1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + n(n+1)u = 0$, for the Legendre polynomials can be written in the form $\alpha_n'' + \alpha_n' \cot\theta = -n(n+1)\alpha_n$. Since $\beta_n = \alpha_n'$ from (10.7), we have $\beta_n' = \alpha_n'' = -\beta_n \cot\theta - n(n+1)\alpha_n$, and $\beta_n'' = -\beta_n' \cot\theta + \beta_n \csc^2\theta - n(n+1)\beta_n$. We list these as

$$(10.8)\quad \begin{cases} \alpha_n' = \beta_n \\ \beta_n' + \beta_n \cot\theta = -n(n+1)\alpha_n \\ \alpha_n'' + \alpha_n' \cot\theta = -n(n+1)\alpha_n \\ \beta_n'' + \beta_n' \cot\theta - \dfrac{\beta_n}{\sin^2\theta} = -n(n+1)\beta_n \end{cases}$$

These will be used to replace derivatives of α_n and β_n by expressions involving α_n and β_n themselves.

For convenience we introduce

$$(10.9)\quad \begin{cases} c_n = b_n' + \frac{2}{r} b_n - \frac{1}{r^2} a_n \\ d_n = a_n' + \frac{2}{r} a_n - n(n+1) b_n \\ e_n = [h(\lambda \mathring{s}_1 - 2\mu) + 2\mu] c_n \end{cases}$$

where

$$h(r) = \frac{1}{\mathring{f}' + \frac{1}{r}\mathring{f}} = \frac{r^3}{2\mathring{a}r^3 - \mathring{b}} .$$

Then

$$\dot{P}^{1j}|_j = \sum [a_n'' + \frac{2}{r} a_n' - \frac{n^2+n+2}{r^2} a_n + \frac{2n(n+1)}{r^2} b_n]\alpha_n$$

$$= \sum [d_n' + n(n+1)c_n]\alpha_n$$

$$\dot{P}^{2j}|_j = \sum [b_n'' + \frac{4}{r} b_n' - \frac{n^2+n-2}{r^2} b_n + \frac{2}{r^3} a_n]\beta_n$$

$$= \sum [c_n' + \frac{2}{r} c_n + \frac{1}{r^2} d_n]\beta_n$$

$$\left.\begin{aligned} \dot{C}^{ij} &= -\dot{C}^{j1} \\ \dot{C}^{12} &= h(\dot{P}^{12} - \dot{P}^{21}) = -h \sum c_n\beta_n \\ \dot{C}^{13} &= h(\dot{P}^{13} - \dot{P}^{31}) = 0 \\ \dot{C}^{23} &= \frac{r}{2\mathring{f}} (\dot{P}^{23} - \dot{P}^{32}) = 0 \end{aligned}\right\} \text{using (4.2) and (4.3)}$$

$$\mathring{s}_1 = 3(\mathring{a}-1)$$

$$\dot{s}_1 = 3\dot{a} + \sum d_n\alpha_n \quad \text{using (4.4)}$$

$$\left.\begin{aligned} \dot{C}^{1j}|_j &= h \sum n(n+1)c_n\alpha_n \\ \dot{C}^{2j}|_j &= \sum (\frac{\partial}{\partial r} + \frac{2}{r})(hc_n)\beta_n \\ \dot{C}^{3j}|_j &= 0 . \end{aligned}\right\} \text{using (5.8)}$$

Substituting into the differential equation (4.6), we obtain two non-trivial equations, and we discover that variables do separate yielding

$$(10.10)\quad \begin{cases} (\lambda + 2\mu)d_n' + n(n+1)e_n = 0 & \text{for } n = 0,1,2,\ldots \\ \dfrac{\lambda + 2\mu}{r^2}\, d_n + e_n' + \dfrac{2}{r}\, e_n = 0 & \text{for } n = 1,2,3,\ldots \end{cases}$$

(since $\beta_o \equiv 0$, we cannot claim the last equation for $n = 0$).

Solving (10.10) we obtain for $n = 1,2,3,\ldots$

$$(10.11)\quad \begin{cases} d_o = \text{constant} \\ d_n = \dfrac{1}{\lambda + 2\mu}\, [-(n+1)A_n r^n + \dfrac{nB_n}{r^{n+1}}] \\ e_n = A_n r^{n-1} + \dfrac{B_n}{r^{n+2}} \\ c_n = \dfrac{1}{h(\lambda \overset{o}{s}_1 - 2\mu) + 2\mu}\, (A_n r^{n-1} + \dfrac{B_n}{r^{n+2}}) . \end{cases}$$

Next solving (10.9) we obtain

$$(10.12)\quad \begin{cases} a_n = \dfrac{1}{2n+1} \Big\{ nr^{n-1} \displaystyle\int \dfrac{(n+1)c_n + \frac{1}{r}\, d_n}{r^{n-2}}\, dr \\ \qquad\qquad - \dfrac{n+1}{r^{n+2}} \displaystyle\int (nc_n - \tfrac{1}{r}\, d_n) r^{n+3}\, dr \Big\} \\ b_n = \dfrac{1}{2n+1} \Big\{ r^{n-2} \displaystyle\int \dfrac{(n+1)c_n + \frac{1}{r}\, d_n}{r^{n-2}}\, dr \\ \qquad\qquad + \dfrac{1}{r^{n+3}} \displaystyle\int (nc_n - \tfrac{1}{r}\, d_n) r^{n+3}\, dr \Big\} \end{cases}$$

for $n = 1,2,3,\ldots$.

Let $H(r) = \dfrac{1}{(\lambda \overset{o}{s}_1 - 2\mu)h(r) + 2\mu}$ and $L_n(r) = \displaystyle\int_{r_1}^{r} H(\rho)\rho^n\, d\rho$.

Then

$$
(10.13)\quad \left\{
\begin{aligned}
a_n &= \Big[\Big(L_1 r^{n-1} - \frac{L_{2n+2}}{r^{n+2}}\Big)\frac{n}{2n+1} - \frac{(n+2)r^{n+1}}{2(2n+3)(\lambda+2\mu)}\Big](n+1)A_n \\
&\quad + \Big[\Big(L_{-2n} r^{n-1} - \frac{L_1}{r^{n+2}}\Big)\frac{n+1}{2n+1} + \frac{n-1}{2(2n-1)(\lambda+2\mu)r^{n}}\Big]nB_n \\
&\quad + \frac{n}{2n+1}\, C_n r^{n-1} - \frac{n+1}{2n+1}\,\frac{D_n}{r^{n+2}} \\
b_n &= \Big[\frac{n+1}{2n+1}\, L_1 r^{n-2} + \frac{n}{2n+1}\,\frac{L_{2n+2}}{r^{n+3}} - \frac{(n+1)r^{n}}{2(2n+3)(\lambda+2\mu)}\Big]A_n \\
&\quad + \Big[\frac{n+1}{2n+1}\, L_{-2n} r^{n-2} + \frac{n}{2n+1}\,\frac{L_1}{r^{n+3}} - \frac{n}{2(2n-1)(\lambda+2\mu)r^{n+1}}\Big]B_n \\
&\quad + \frac{C_n r^{n-2}}{2n+1} + \frac{D_n}{(2n+1)r^{n+3}}
\end{aligned}
\right.
$$

for $n = 1,2,3,\ldots$ where C_n and D_n are constants of integration.

Next we consider the perturbed boundary conditions. From (10.2) for $r = r_2$ we obtain

$$
\begin{aligned}
\dot{Q}^{i1} &= -\frac{\varepsilon^{ijk}}{r^2 \sin\theta}\,(\dot{T}\,\mathring{P}_{j2}\mathring{P}_{k3} + \mathring{T}\,\dot{P}_{j2}\mathring{P}_{k3} + \mathring{T}\,\mathring{P}_{j2}\dot{P}_{k3}) \\
&= -\frac{\dot{T}}{r^2\sin\theta}\,\varepsilon^{i23}\,\mathring{P}_{22}\mathring{P}_{33} - \frac{\mathring{T}}{r^2\sin\theta}\,(\varepsilon^{ij3}\,\dot{P}_{j2}\mathring{P}_{33} + \varepsilon^{i2k}\,\mathring{P}_{22}\dot{P}_{k3})\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\dot{Q}^{11} &= -\frac{\dot{T}\mathring{f}^2}{r^2} - \frac{\mathring{T}\mathring{f}}{r^3}\,\dot{P}_{22} - \frac{\mathring{T}\mathring{f}}{r^3\sin^2\theta}\,\dot{P}_{33} \\
&= -\frac{\dot{T}\mathring{f}^2}{r^2} - \frac{2\mathring{T}\mathring{f}\dot{f}}{r^2} - \frac{\mathring{T}\mathring{f}}{r^2}\sum\Big[b_n(\beta_n' + \beta_n\cot\theta) + \frac{2}{r}\,a_n\alpha_n\Big] \\
&= -\Big(\frac{\dot{T}f^2}{r^2}\Big) - \frac{\mathring{T}\mathring{f}}{r}\sum\Big[\frac{2}{r}\,a_n - n(n+1)b_n\Big]\alpha_n\,.
\end{aligned}
$$

Similarly

$\dot{Q}^{21} = \frac{\overset{\circ\circ}{T f}}{r^3} \sum (a_n - r b_n)\beta_n$ and $\dot{Q}^{31} = 0$ for $r = r_2$.

From (4.5) we have

$$\dot{Q}^{11} = (3\lambda + 2\mu)\dot{a} - \frac{4\mu\dot{b}}{r^3} + \sum (\lambda d_n + 2\mu a_n')\alpha_n ,$$

$$\dot{Q}^{21} = \sum [2\mu(b_n' + \frac{1}{r} b_n) + (\lambda\overset{\circ}{s}_1 - 2\mu)hc_n]\beta_n, \text{ and } \dot{Q}^{31} = 0 .$$

From (10.3) we have $(3\lambda + 2\mu)\dot{a} - \frac{4\mu\dot{b}}{r^3} = -(\frac{\dot{T}f^2}{r^2})$ for $r = r_2$.

Thus from the above expressions of $\dot{Q}^{i1}$ we have for $r = r_2$

$$(10.14)\quad \begin{cases} (\lambda + 2\mu)d_n + (\frac{\overset{\circ\circ}{Tf}}{r} - 2\mu)[\frac{2}{r} a_n - n(n+1)b_n] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n = 0,1,2,\ldots \\ e_n - (\frac{\overset{\circ\circ}{Tf}}{r} - 2\mu)(\frac{1}{r^2} a_n - \frac{1}{r} b_n) = 0 \quad \text{for } n = 1,2,3,\ldots . \end{cases}$$

Similarly for $r = r_1$

$$(10.15)\quad \begin{cases} (\lambda + 2\mu)d_n - 2\mu[\frac{2}{r} a_n - n(n+1)b_n] = 0 \quad \text{for } n = 0,1,2,\ldots \\ e_n + 2\mu(\frac{1}{r^2} a_n - \frac{1}{r} b_n) = 0 \qquad\qquad \text{for } n = 1,2,3,\ldots . \end{cases}$$

For $n = 1,2,3,\ldots$, equations (10.14) and (10.15) are four linear homogeneous equations in A_n, B_n, C_n, and D_n. Setting the determinant of coefficients equal to zero, we obtain an equation for the critical values of T. It is now clear how the critical values of T can be computed numerically; however, we have not carried out such calculations.

Lubkin [3] finds the lowest critical load for nearly solid spheres and for thin spherical shells, and he observes that the result for thin shells agrees with that found by the usual thin shell procedures. The formulation given here can be used relatively conveniently to calculate the critical loads for any hollow sphere and any of the modes of buckling considered here.

Bibliography

[1] Green, A. E., and Zerna, W., Theoretical Elasticity, Clarendon Press, Oxford, 1954.

[2] Murnaghan, F. D., Finite Deformations of an Elastic Solid, John Wiley and Sons, New York, 1951.

[3] Lubkin, S., Determination of Buckling Criteria by Minimization of Total Energy, New York University, Courant Institute of Mathematical Sciences, Report IMM-NYU 241, 1957.

[4] John, F., Plane Strain Problems for a Perfectly Elastic Material of Harmonic Type, Comm. Pure Appl. Math., Vol. XIII, No. 2, 1960, pp. 239-296.

[5] John F., On Finite Deformations of an Elastic Isotropic Material, New York University, Courant Institute of Mathematical Sciences, Report IMM-NYU 250, 1958.

[6] John, F., Rotation and Strain, Comm. Pure Appl. Math., Vol. XIV, No. 3, 1961, pp. 391-413.

[7] Sensenig, C., The Buckling of a Thick Circular Plate Using a Non-Linear Theory, New York University, Courant Institute of Mathematical Sciences, Report IMM-NYU 262, 1959.

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USN and Nav. Inspec. of Ordnance
General Dynamics Corp.,
Electr. Boat Div.
Groton, Connecticut (1)

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Newport News Shipbuilding
and Dry Dock Co.
Newport News, Virginia (1)

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USN and Nav. Inspec. of Ordnance
Ingalls Shipbuilding Corp.
Pascagoula, Mississippi (1)

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MIT, Cambridge 39, Mass. (1)

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Webb Inst. of Nav. Arch.
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Washington 25, D. C. (1)

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Attn: Mechanics Div. (1)

U.S. Atomic Energy Commission
Washington 25, D. C.
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